The topography of gravitational graphs

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Reminder on graphs
A *non oriented graph* $G = [\mathcal{N}, \mathcal{E}]$: $\mathcal{N}$ = vertices or nodes; $\mathcal{E}$ = of edges; an edge being a pair of vertices.

The nodes are designated with small letters: $p, q, r...$

The edge linking the nodes $p$ and $q$ is designated by $e_{pq}$.

The subgraph spanning a set $A \subset \mathcal{N}$ is the graph $G_A = [A, \mathcal{E}_A]$, where $\mathcal{E}_A$ are the edges linking two nodes of $A$.

The partial graph $G' = [\mathcal{N}, \mathcal{E}']$ has the same set of nodes as $G$ but only a subset $\mathcal{E}' \subset \mathcal{E}$ of edges.

A *path*, $\pi$, is a sequence of vertices and edges: $\pi$ starts with a vertex, say $p$, followed by an edge $e_{pq}$, incident to $p$, followed by the other endpoint $q$ of $e_{pq}$, and so on.
Let $T$ be a totally ordered set. Nodes and/or edges may be weighted with the functions:

- $F_n = \text{Fun}(\mathcal{N}, T)$: the functions defined on the nodes $\mathcal{N}$ with value in $T$; the function $\nu \in F_n$ takes the weight $\nu_p$ on the node $p$
- $F_e = \text{Fun}(\mathcal{E}, T)$: the functions defined on the edges $\mathcal{E}$ with value in $T$; the function $\eta \in F_e$ takes the value $\eta_{pq}$ on the edge $e_{pq}$

Notations:
A graphs $G = [\mathcal{N}, \mathcal{E}]$ holding
- only node weights $\nu \in F_n$ is designated by $G(\nu, \text{nil})$
- only edge weights $\eta \in F_e$ is designated by $G(\text{nil}, \eta)$
- node and edge weights is designated by $G(\nu, \eta)$
The topography of weighted graphs
# Flowing edges and flowing paths

<table>
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<td>$G(\nu, \text{nil})$ : An edge $(p, q)$ $e_{pq}$ is a <strong>flowing edge</strong> of its extremity $p$ iff $\nu_p \geq \nu_q$.</td>
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→ $p, q, s$ are 3 successive nodes on a flowing path, then $\nu_p \geq \nu_q \geq \nu_s$

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<td>$G(\text{nil}, \eta)$ : An edge $(p, q)$ is a <strong>flowing edge</strong> of the node $p$, if it is one of the lowest adjacent edges of $p$.</td>
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→ the weights of the successive edges in a flowing path are never increasing

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Flowing paths

**Definition**

A path \( p = e_{pq} - q = e_{qs} - s \ldots \) is a **flowing path** if each node except the last one is followed by one of its flowing edges.

For a node weighted graph: the weights of the nodes along a flowing path are never increasing: if \( p, q, s \) are 3 successive nodes on a flowing path, then \( \nu_p \geq \nu_q \geq \nu_s \).

For an edge weighted graph: the weights of the successive edges in a flowing path are never increasing (if \( p, q \) and \( r \) are three successive nodes of a flowing path, then \( e_{qr} \) is one of the lowest edges of \( q \) and thus \( e_{pq} \geq e_{qr} \)).
In a node weighted graph: each edge is a flowing edge of at least one extremity of this edge.

For the edge \( e_{st} \), either \( \nu_s \geq \nu_t \) and \( e_{st} \) is a flowing edge of \( s \), or \( \nu_t \geq \nu_s \) and \( e_{st} \) is a flowing edge of \( t \).

The edge \( e_{st} \) is a flowing edge of both its extremities if \( \nu_s = \nu_t \).

The nodes without lower neighbors are not the origin of a flowing edge, they are isolated regional minima.
A selfloop is added to each isolated regional minimum. A drop of water arriving at such a node cycles around it, with no possibility to leave it. We obtain the flowing graph $\bigcirc G(\nu,\text{nil})$, in which each node is the origin of a flowing edge and each edge is a flowing edge of one of its extremities.
In a connected graph, each node has one or several neighboring edges; those with minimal weight are the flowing edges for this node. **Thus, each node is the extremity of at least one flowing edge.** However, **not each edge is a flowing edge**: If the extremities of an edge \( e_{st} \) have adjacent edges with weights lower than \( \eta_{st} \), then \( e_{st} \) is not a flowing edge.

Such an edge never belongs to the trajectory of a drop of water freely circulating from node to node.
If an edge of $G(nil, \eta)$ is not the lowest edge of one of its extremities, it may be cut without disturbing the trajectories of a drop of water. We obtain the flowing graph $\downarrow G(nil, \eta)$, in which each node is the origin of a flowing edge and each edge is a flowing edge of one of its extremities.
From flowing graphs to flowing digraphs
Flowing graph $G$: each node is the origin of a flowing edge and each edge is a flowing edge of one of its extremities

$G \longrightarrow \vec{G} : G$ and $\vec{G}$ have the same nodes; the arcs of $\vec{G}$ correspond to the flowing edges of $G$:

- for $G(\nu, nil)$: \[ \{ p \rightarrow q \text{ in } \vec{G} \} \iff \{ \nu_p \geq \nu_q \text{ in } G \} \]

- for $G(nil, \eta)$: \[ \{ p \rightarrow q \text{ in } \vec{G} \} \iff \{ e_{pq} \text{ is a flowing edge of } p \text{ in } G(nil, \eta) \} \iff \{ e_{pq} \text{ is a flowing edge of } p \text{ in } \downarrow G(nil, \eta) \} \]
Associating a flowing digraph to a node weighted flowing graph

A: a node weighted graph $G(v, nil)$

B: the associated flowing digraph $\mathcal{G}(v, nil)$
The flowing paths of a flowing graph or digraph

A: a node weighted graph $G(\nu, \text{nil})$
B: the associated flowing digraph $\overrightarrow{G}(\nu, \text{nil})$

To each flowing path in the flowing graph corresponds an oriented path in the flowing digraph.
Two oriented paths with the same origin towards 2 distinct regional minima.
An isolated regional minimum has no lower or equal neighbor. For the elegance of the theoretical developments below we add a loop edge linking the isolated regional minimum with itself. However, in the algorithms developed for implementing the theory, this step is often not needed.
Associating a flowing digraph to an edge weighted flowing graph

A: an edge weighted flowing graph \( G(nil, \eta) \)

B: the associated flowing digraph \( \vec{G}(nil, \eta) \)
The flowing paths of a flowing graph or digraph

A: an edge weighted flowing graph $G(nil, \eta)$
B: the associated flowing digraph $\overrightarrow{G}(nil, \eta)$

To each flowing path in the flowing graph corresponds an oriented path in the flowing digraph.
The flowing paths of an edge weighted flowing digraph

Two oriented paths with the same origin towards 2 distinct regional minima.
Flat zones
Notations and relations

- $\overrightarrow{pq}$ designates an oriented path between $p$ and $q$ in a digraph;
- the relation $\overleftarrow{pq}$ indicates that there exists an oriented path $\overrightarrow{pq}$ connecting $p$ and $q$.
- $\overrightarrow{pq}$ designates a path between $p$ and $q$ in a digraph in which all arcs are bidirectional arcs;
- the relation $\overleftrightarrow{pq}$ indicates that there exists a path $\overrightarrow{pq}$, connecting $p$ and $q$. 
Oriented paths and bi-directional paths

If we suppose that the relations \( \rhd \) and \( \iff \) are reflexive, i.e. \( p \rhd p \) and \( p \iff p \), then:

- the relation \( p \rhd q \), i.e. there exists a path \( p \rhd q \) connecting \( p \) and \( q \), is a preorder relation.
- the relation \( \{ p \rhd q \text{ and } q \rhd p \} \) is an equivalence relation. Its equivalence classes are the flat zones of the oriented graph \( \overrightarrow{G} \).
- the relation \( p \iff q \) i.e. there exists a path \( p \iff q \), connecting \( p \) and \( q \) and containing only bidirectional arcs, is also an equivalence relation. Its equivalence classes are the superflat zones of the non oriented graph \( G \).

As \( \{ p \rhd q \text{ and } q \rhd p \} \iff \{ p \iff q \} \), the superflat zones are sub-sets of the flat zones. They are identical only in the case of gravitational digraphs.
A counter example

We define: for $p$, $q$ neighbors, \( \{ p \rightarrow q \} \iff \{ \nu_p + \lambda \geq \nu_q \} \), implying \( \{ p \leftarrow q \} \iff \{ \nu_p - \lambda \leq \nu_q \leq \nu_p + \lambda \} \).

A: a digraph, which is not a gravitational graph ($\lambda = 1$)

B: The equivalence classes for the relation $pRq = \{ p \searrow q \text{ and } q \searrow p \}$: flat zones

C: The partial non oriented graph $\overline{G}$, obtained by replacing $(p \leftrightarrow q)$ by $(p - q)$.

D: The equivalence classes of $\overrightarrow{G}$ for the relation $(p \leftrightarrow q)$: superflat zones
Upstream and downstream propagation
Upstream propagation in the digraph associated to a node weighted graph

A node weighted graph and its associated gravitational graph. A red is highlighted and its upstream (in blue) extracted.
Upstream propagation in the digraph associated to an edge weighted graph

An edge weighted graph and its associated gravitational graph. A red node is highlighted and its upstream (in blue) extracted.

An edge weighted graph and its associated gravitational graph. A red node is highlighted and its upstream (in blue) extracted.
Downstream propagation in the digraph associated to a node weighted graph

Diagram A and B
Downstream propagation in the digraph associated to an edge weighted graph
Gravitational digraphs
Ever descending stairs are impossible in node or edge weighted digraphs
Ever descending stairs are impossible in node weighted digraphs: Consider a flowing path, i.e. a never ascending path $\pi_{pq} = p \cdots q$ between $p$ and $q$, implying $\nu_p \geq \nu_q$. If there exists a second never ascending path $\sigma_{qp} = q \cdots p$ between $q$ and $p$, then $\nu_q \geq \nu_p$. Together, these two relations show that $\nu_p \geq \nu_q \geq \nu_p$, implying that all nodes along both paths $p \cdots q$ and $q \cdots p$ have the same weight. These paths are in fact bidirectional graphs $\pi_{pq} = \pi_{qp}$ and $\sigma_{pq} = \sigma_{qp}$.
No perpetuum mobile in edge weighted digraphs

Ever descending stairs are impossible in edge weighted digraphs: If there exist two flowing paths $\overrightarrow{\pi}_{pq} = p \cdots q$ between $p$ and $q$, and $\overrightarrow{\sigma}_{qp} = q \cdots p$ between $q$ and $p$, the concatenation between both paths $\overrightarrow{\pi}_{pq} \circlearrowright \overrightarrow{\sigma}_{qp}$ is a loop along which the edge weights are never increasing. The first and the last edge of the loop verify: $\eta_{pt} \geq \eta_{sp}$

\[ p \rightarrow t \rightarrow \cdots \rightarrow \Diamond \rightarrow q \]
\[ \uparrow \]
\[ s \leftarrow \Diamond \leftarrow \cdots \leftarrow \Diamond \]

On the other hand, the path $s \rightarrow p \rightarrow t$ also is a non increasing paths and the first and last edge verify: $\eta_{sp} \geq \eta_{pt}$. Hence $\eta_{sp} = \eta_{pt}$ and the weights of all edges along the loop $\overrightarrow{\pi}_{pq} \circlearrowright \overrightarrow{\sigma}_{qp}$ are equal. These paths are in fact bidirectional graphs $\overrightarrow{\pi}_{pq} = \overleftarrow{\pi}_{pq}$ and $\overrightarrow{\sigma}_{pq} = \overleftarrow{\sigma}_{pq}$.
The existence of a never ascending path $p \cdot \cdots q$ in $G(\nu, nil)$ is expressed by $p \downarrow q$ in the digraph. Thus for both node and edge weighted digraphs we have: \[ \{ p \downarrow q \text{ and } q \downarrow p \} \Rightarrow \{ p \leftrightarrow q \} \]. The inverse implication is always true.

We call "gravitational digraphs", the digraphs verifying \[ \{ p \downarrow q \text{ and } q \downarrow p \} \Leftrightarrow \{ p \leftrightarrow q \} \]

We now study the properties of the gravitational digraphs, independently of any node or edge weights.
Black holes and catchment zones
The gravitational attraction of a black hole is so high that it catches everything around it and not even the light gets out. A node without any outgoing arc is an isolated black hole. Non isolated black holes are flat zones without outgoing arcs:

**Definition**

A black hole $M$ of a gravitational graph is defined by the following equivalence: $p \in M \Leftrightarrow \{ \forall q : p \not\rightarrow q \Rightarrow q \not\rightarrow p \}$

If there exists no node $q$ such that $p \not\rightarrow q$, then $p$ is an isolated black hole. If on the contrary, $p \not\rightarrow q$ for a given node $q$, then $q \not\rightarrow p$, implying $p \leftrightarrow q$, i.e. $p$ and $q$ belong to the same flat zone: there is no way to escape from this flat zone through an outgoing arc.

Suppose that for $p \in M : p \rightarrow s$ for a particular node $s$, then:

$p \rightarrow s \Rightarrow p \not\rightarrow s \Rightarrow s \not\rightarrow p \Rightarrow p \leftrightarrow s$, showing that $s$ also belongs to the same flat zone.
Catchment zones

Each black hole is surrounded by a catchment zone.

**Definition**

The catchment zone of a black hole \( M \) is the set of nodes at the origin of an oriented path leading inside \( M \): 

\[
\text{CZ}(M) = \{ p \in \mathcal{N} \mid p \xrightarrow{\text{out}} q, \quad q \in M \}
\]

If there exists \( q \in M \), such that \( p \xrightarrow{\text{out}} q \), then \( p \xrightarrow{\text{out}} s \) for each \( s \in M \), as \( q \xleftrightarrow{\text{in}} s \). The following theorem shows that each node of a gravitational graph belongs to the attraction zone of at least one black hole. These attraction zones may overlap and a node belong to the attraction zones of several black holes.

**Theorem**

*Each node of a gravitational graph belongs to the attraction zone of at least one black hole.*
Proof of the theorem

If \( p \), a node of a gravitational graph \( \overrightarrow{G} \), belongs to a black hole, the theorem is verified.

If not: \( \exists \ q_1 : p \not\prec q_1 \) and \( q_1 \not\succ p \)

Again there are two possibilities :

a) \( q_1 \) belongs to a black hole, the theorem is verified

b) \( q_1 \) does not belong to a black hole and there exists \( q_2 \), such that \( q_1 \not\prec q_2 \) and \( q_2 \not\succ q_1 \). Necessarily \( q_2 \not\succ p \) otherwise there exists a cycle verifying \( p \not\prec q_1 \not\prec q_2 \not\prec p \), implying \( q_1 \iff q_2 \), and in particular \( q_2 \not\prec q_1 \), which is contrary to the hypothesis. The relation \( q_2 \not\succ p \) also shows that \( q_2 \neq p \).

The same reasoning may be applied again and again, producing a series of nodes \( q_i \), verifying \( p \not\prec q_1 \not\prec q_2 \not\prec ... q_k \) and \( q_k \not\succ q_{k-1} \not\succ ... q_1 \not\succ p \) all distinct from each other. As the number of nodes of \( \mathcal{N} \) is finite, the series \( q_i \) converges to a node belonging to a black hole of the graph.
An algorithm for detecting the black holes

If \( p \) does not belong to a black hole, there exists an oriented path \( p \rightarrow q \) connecting \( p \) with a node \( q \) inside a black hole. There exists a couple of nodes \( s, u \), along this path, verifying \( s \rightarrow u \) and \( u \rightarrow s \) otherwise we have \( p \leftrightarrow q \), and \( p \) would belong to the same black hole as the node \( q \). We may suppose that \((s, u)\) is the first couple of nodes if one follows this path upstream, verifying \( s \rightarrow u \) and \( u \rightarrow s \). Then \( p \) belongs to the upstream of the node \( s \), i.e. to \( \{ t \in N \mid t \downarrow s \} \).

**Lemma**

*The non black holes are the upstream of all nodes \( s \) having a neighbor \( u \) verifying \( s \rightarrow u \) and \( u \rightarrow s \). The black holes are obtained by taking their complement.*
An algorithm for detecting the black holes

The red nodes: $s \rightarrow u$ and $u \rightarrow s$. The blue nodes: upstream of the red nodes. Together, red and blue nodes are the non black holes. The black holes, their complement are labeled in fig.D.
An algorithm for detecting the black holes on an initially edge weighted graph
Extracting the catchment zone containing a marked node.

A: A gravitational graph with its labeled regional minima. B: marking a node $p$. C: Downstream of the marked node $p$. D: Upstream of the downstream $CZ(p) = \uparrow \downarrow p$
C,D: 2 overlapping catchment zones. E: Upstream propagation of the cyan and violet label. The violet node pointing towards a cyan label belongs to both catchment zones. F: its upstream constitutes the overlapping of both catchment zones.
Extracting overlappings of catchment zones

We consider a labeled flowing digraph $\vec{G} (\text{nil}, \text{nil}, \lambda)$ in which the labels form a partial partition, result of the upstream label propagation of the labels of some marked nodes. If two node $p, q$ verifying $(p \rightarrow q)$ hold distinct labels, the node $p$ belongs to two distinct catchment zones. The label of $q$ could not be propagated to the node $p$, as the label of $q$ has a lower priority than the label of $p$. Such nodes are called "overlapping seeds", forming a set $S$ defined by:

$$S = \{ p \mid p, q \in N : (p \rightarrow q) \text{ and } (\lambda_p \neq \lambda_q) \}$$
C,D: 2 overlapping catchment zones. E: Upstream propagation of the yellow and violet labels. A violet node pointing towards a yellow label belongs to both catchment zones. F: its upstream constitutes the overlapping of both catchment zones.
The restricted catchment zones.

A: A gravitational digraph; B: The catchment zone of the black hole labeled in violet.
C: The violet nodes pointing towards a node with a distinct label also belong to other catchment zones, as does their upstream in C
D: The remaining violet nodes constitute a restricted catchment zone.
Authorized prunings

Each node belongs to the attraction zone of at least a black hole, sometimes of several. In order to reduce the attraction zone of a black hole, it is sufficient to reduce the number of oriented paths leading to it. This may be done by suppressing arcs in the graph, without modifying the black holes themselves. If $e_{pq}$ is an arc which is suppressed, then all paths passing through this arc are broken. Such an operation suppressing arcs is called pruning the graph. By pruning the graph the attraction zone of the black holes will be reduced. Authorized prunings leave the black holes unchanged but narrow their catchment zones.

**Definition**

In an authorized pruning, the arc $p \rightarrow q$ may be suppressed if after pruning there remains a node $u$ such that $p \rightarrow u$ and $u \rightarrow p$.

**Lemma**

An authorized pruning preserves the black holes.
Authorized prunings

A: a black hole in violet ; B: its catchment zone
C: authorized pruning suppressing an arrow, leaving the black holes unchanged ; D: restricted catchment zone
Back to the node or edge weighted graphs
Gravittational graphs derived from node weighted graphs
It is defined by: for $p, q$ neighbors $\{p \to q\} \iff \{\nu_p \geq \nu_q\}$. Thus $\{p \leftrightarrow q\} \iff \{\nu_p = \nu_q\}$. Its flat zones are maximal connected components of the graph, surrounded by nodes with a different weight. If no neighboring node has a lower altitude, a flat zone is a regional minimum.

The black holes are thus the regional minima. They are either isolated regional minima, i.e. nodes with no lower or equal neighbors. Or they are extended flat zones without lower neighbors.
The node weighted graph and the associated gravitational digraph have the same regional minima.
A pruning keeping only the arcs towards the lowest neighbors

In an authorized pruning, the arc $p \rightarrow q$ may be suppressed if after pruning there remains a node $u$ such that $p \rightarrow u$ and $u \rightarrow p$. The pruning which leaves for each node only the arrows towards its lowest neighbors is an authorized pruning. If the node $q$ is not one of the smallest neighbors of $p$, then there exists at least a lowest neighbor $u$ such that $p \rightarrow u$ and $u \rightarrow p$ and the arc $p \rightarrow q$ may be suppressed.

Submitted to an authorized pruning, the resulting graph has the same regional minima and smaller catchment zones. The flat zones also are smaller.
A pruning keeping only the arcs towards the lowest neighbors
Gravitational graphs derived from edge weighted graphs
In $G(nil, \eta)$, an edge $(p, q)$ is a flowing edge of the node $p$, if it is one of the lowest adjacent edges of $p$. The arcs of the digraph are defined by: for $p, q$ neighbors \( \{p \to q\} \iff e_{pq} \) is a flowing edge of $p$.

The flat zones are the equivalence classes of the equivalence relation $q \leftrightarrow s$. The flat zones are thus maximal connected components of nodes such that two neighboring nodes are connected by an edge which is the lowest edge of both its extremities.

The regional minima are flat zones without outgoing arcs. Isolated regional minima do not exist.
Flat zones and black holes of a gravitational digraph

A, B: an edge weighted graph and its flowing digraph. C, D: the flat zones obtained by replacing the double arrows by edges.

E, F: the flat zones without outgoing arcs are the black holes of the digraph and the regional minima of the edge weighted graph.