A hydostatic model representing edge or node weighted graphs

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A hydrostatic model
A non oriented graph $G = [\mathcal{N}, \mathcal{E}]$: $\mathcal{N}$ = vertices or nodes; $\mathcal{E}$ = of edges; an edge being a pair of vertices.

Notations: small letters for the nodes $p, q, r...$: $e_{pq}$ the edge connecting $p$ and $q$.

Let $\mathcal{T}$ be a totally ordered set. Nodes and/or edges may be weighted with the functions:

- $\mathcal{F}_n = \text{Fun}(\mathcal{N}, \mathcal{T})$: the functions defined on the nodes $\mathcal{N}$ with value in $\mathcal{T}$; the function $\nu \in \mathcal{F}_n$ takes the weight $\nu_p$ on the node $p$.

- $\mathcal{F}_e = \text{Fun}(\mathcal{E}, \mathcal{T})$: the functions defined on the edges $\mathcal{E}$ with value in $\mathcal{T}$; the function $\eta \in \mathcal{F}_e$ takes the value $\eta_{pq}$ on the edge $e_{pq}$. 
A tank network for representing a node or edge weighted graph

We consider a node and edge weighted graph \( G(\nu, \eta) \). The nodes are represented as vertical tanks of infinite height and depth. The weight \( \nu_i \) represents the level of water in the tank \( i \), equal to \(-\infty\) if no water is present. Two neighboring tanks \( i \) and \( j \) are linked by a pipe at an altitude \( \eta_{ij} \) equal to the weight of the edge. We associate like that to each node and edge weighted graph a tank network. The distribution of water in the tanks is at equilibrium for the tanks \((a...f)\) but not for the tanks \((g, h)\) or \((i, j)\).
The hydrostatic equilibrium

For arbitrary weights of nodes and edges, the distribution of water does not follow the laws of hydrostatics. To be at equilibrium, edge and node weights have to be coupled: the level $\nu_i$ in the tank $i$ cannot be higher than the level $\nu_j$, unless $\eta_{ij} \geq \nu_i$.

**Definition**

The distribution $\nu$ of water in the tanks of the graph $G = [E,N]$ is a flooding of this graph, i.e. is a stable distribution of fluid if it verifies the criterion hydrostat1:

for any couple of neighboring nodes $(p, q): (\nu_p > \nu_q \Rightarrow \eta_{pq} \geq \nu_p)$
A tank network for representing a node or edge weighted graph

Tank and pipe network:
- $a$ and $b$ form a regional minimum with $\nu_a = \nu_b = \lambda$; $\eta_{ab} < \lambda$; $\eta_{bc} > \lambda$
- $b$ and $c$ have unequal levels but are separated by a higher pipe: $\nu_c > \nu_b$ and $\eta_{bc} > \nu_c$
- $d$ and $e$ form a full lake, reaching the level of its lowest exhaust pipe $\eta_{cd}: \nu_d > \nu_c$ and $\eta_{cd} \geq \nu_d$
- $e$ and $f$ have the same level, without forming a lake, as they are linked by a higher pipe.

Non equilibrium for $(g, h)$ and $(i, j): \nu_h > \nu_g$ but $\eta_{gh} < \nu_h$. Similarly: $\nu_i > \nu_j$ but $\eta_{ii} < \nu_i$.
The following equivalences yield other useful criteria for recognizing flood distributions on tank networks:

\[(\nu_p > \nu_q \implies \eta_{pq} \geq \nu_p) \iff (\text{not} (\nu_p > \nu_q) \text{ or } \eta_{pq} \geq \nu_p) \iff\\(\nu_p \leq \nu_q \text{ or } \nu_p \leq \eta_{pq}) \iff (\nu_p \leq \nu_q \vee \eta_{pq})\] (criterion hydrostat2)
The criterion for the hydrostatic equilibrium:

\[(\nu_p > \nu_q \Rightarrow \eta_{pq} \geq \nu_p)\]

\[(\nu_p > \nu_q \Rightarrow \eta_{pq} \geq \nu_p) \iff (\eta_{pq} < \nu_p \Rightarrow \nu_p \leq \nu_q)\]

If follows from the second implication:

\[\eta_{pq} < \nu_p \Rightarrow \nu_p \leq \nu_q \Rightarrow \eta_{pq} < \nu_p \leq \nu_q \Rightarrow \nu_q \leq \nu_p\]

Putting everything together we get \(\eta_{pq} < \nu_p \Rightarrow \nu_p = \nu_q\) showing that if the level \(\nu_p\) in the tank \(p\) is higher than the pipe \(\eta_{pq}\), then \(\nu_p = \nu_q\), which is indeed what happens in case of hydrostatic equilibrium.
Critical fillings of a tank network
Perturbating a tank network

Consider a tank network representing a node and edge weighted graph. If its filling is at hydrostatic equilibrium:

- Adding a drop of water to a tank may produce an overflow towards other pipes.
- Similarly, lowering the level of a pipe may produce a leakage through this pipe from one tank to another.
The filling of the tank network is critical if any modification lets water entering into a pipe:

- the filling is "node-critical" if adding a drop of water in a tank produces an overflow in an adjacent pipe.
- the filling is "edge-critical" if lowering the edge level of a pipe produces an overflow through this pipe.

A tank network will be said fully critical if it is both node and edge critical. Such tanks are remarkable, as they offer a perfect coupling between the edge weights and the node weights, which may be derived from each other.
Critical fillings of the tank network of a node weighted graph

1: tank network associated to a node weighted graph: each tank is filled up to the level of its weight
2: an edge critical filling: pipes link neighboring tanks at the lowest level possible, without provoking an overflow into a pipe. Any lowering of a pipe would provoke an overflow.
3: the filling is not node critical: adding a drop of water in a tank does not provoke an overflow in tank $c$, corresponding to an isolated regional minimum
1: tank network associated to a node weighted graph: each tank is filled up to the level of its weight
2: pipes are introduced in order to produce an edge critical filling of the tanks: pipes link neighboring tanks at the lowest level possible, without provoking circulation of water in the pipe. Any lowering of a pipe would provoke an overflow. A "auto-pipe" connecting the tank \( c \) with itself is added, at the level of its filling.
3: the filling is now also node critical: adding a drop of water in a tank does provoke an overflow in each tank, including \( c \).
Critical fillings of the tank network of an edge weighted graph

1: tank network associated to an edge weighted graph: each edge of the graph is replaced by a pipe at the same altitude as its weight.
2: a node critical filling: each tank is filled at the highest possible level without provoking an overflow. Adding a drop of water in any tank provokes an overflow.
3: the filling is not edge critical: by slightly lowering the edge \((D, E)\), it is not filled by water. This edge is not the lowest of any of its extremities.
Successive critical fillings (nod weighted graph)

1: tank network associated to a node weighted graph: each tank is filled up to the level of its weight
2,3: edge critical filling: pipes link neighboring tanks at the lowest level possible, without provoking an overflow into a pipe. Any lowering of a pipe would provoke an overflow.
4: we empty the tanks (3)
5: a new edge critical filling of the tanks (4)
6: Comparison between the node weights in (1) and (5,6): the isolated regional minimum has been filled up to the level of its lowest neighbor.
Critical fillings of the tank network of a node weighted graph

1,2,3: edge critical filling of the tanks associated to a node weighted graph

4,5,6: The graph structure is edited, by connecting the isolated regional minimum tank c by an "auto-pipe" with itself. The resulting filling is both edge and node critical.
Successive critical fillings (node weighted graph)

1: tank network associated to a node weighted graph: each tank is filled up to the level of its weight

2,3: The graph structure is edited, by connecting the isolated regional minimum tank \( c \) by an "auto-pipe" with itself. The resulting filling is both edge and node critical.
Successive critical fillings (node weighted graph)

4: we empty the tanks (3)
5: a node critical filling of the tanks (4) produces the same flood distribution as in (1).
6: Comparison between the node weights in (1) and (5,6): the levels are identical
Successive critical fillings (edge weighted graph)

1: tank network associated to an edge weighted graph: each edge of the graph is replaced by a pipe at the same altitude as its weight.
2,3: a node critical filling: each tank is filled at the highest possible level without provoking an overflow. Adding a drop of water in any tank provokes an overflow.
4: The pipes are suppressed, the levels of water in the tanks remaining unchanged.
Successive critical fillings (edge weighted graph)

4: The pipes are suppressed, the levels of water in the tanks remaining unchanged.
5: Pipes are introduced in order to produce an edge critical filling of (4). The level of the pipe \((D, E)\) has been lowered.
6: Comparison between the pipe levels in (1) and (6): the pipe which is not the lowest neighbor of one of its extremities has been lowered. Its level is equal to the highest adjacent flowing pipe \((E, F)\).
Critical fillings of the tank network of an edge weighted graph

1, 2, 3: node critical filling of the tank network associated to an edge weighted graph. The filling is not edge critical, the edge \((D, E)\) is not filled by water when it is lowered.

4, 5, 6: the pipe \((D, E)\) of (1) has been suppressed. The node critical filling of the new tank network is both edge and node critical.
1: tank network of an edge weighted graph
2: the edge (D,E) which is not the lowest edge of one of its extremities is suppressed, creating 2 sub networks
3: the node critical filling of (3) is also edge critical
Successive fillings of the tank network of an edge weighted graph

3: the node critical filling of (3) is also edge critical
4: the pipes are suppressed, without changing the levels of water in the tanks
5: pipes are introduced in order to produce an edge critical filling.
6: the weights of the pipes in (1) and (6) are identical
Morphological interpretation
Critical fillings of the tank network of a node weighted graph

1: each tank represents a node of the graph, filled up to a level equal to the weight of this node.
2: pipes link neighboring tanks at the lowest level possible, without provoking circulation of water in the pipe.
The level of the pipe \((p, q)\) is \(\eta_{pq} = \nu_p \lor \nu_q = (\delta_{en} \nu)_{pq}\). The hydrostatic equilibrium is verified: \(\eta_{pq}\) is the lowest level verifying both \((\nu_p \leq \nu_q \lor \eta_{pq})\) and \((\nu_q \leq \nu_p \lor \eta_{pq})\).
3: the filling is not node critical: adding a drop of water in a tank does not provoke an overflow into tank \(c\), an isolated regional minimum.
Critical fillings of the tank network of an edge weighted graph

1: tank network associated to an edge weighted graph: each edge of the graph is replaced by a pipe at the same altitude as its weight.

2: a node critical filling: each tank is filled at the highest possible level without provoking an overflow. Therefore the highest level $\nu_p$ of water in a tank $p$ has to be equal to the level of the lowest adjacent pipe:

$$\nu_p = \bigwedge \eta_{ps} = (\epsilon_{ne}\eta)_p.$$  

The hydrostatic equilibrium is verified: If $p$ and $q$ are two neighboring nodes

$$\nu_p = \bigwedge \eta_{ps} \leq \eta_{pq}$$

and

$$\nu_q = \bigwedge \eta_{qs} \leq \eta_{pq} \quad \text{hence} \quad \nu_p \leq \nu_q \lor \eta_{pq} \quad \text{and} \quad \nu_q \leq \nu_p \lor \eta_{pq}.$$
Critical fillings of the tank network of an edge weighted graph

3: the filling is not edge critical: by slightly lowering the edge $(D, E)$, it is not filled by water. This edge is not the lowest of any of its extremities.
1: tank network associated to a node weighted graph: each tank is filled up to the level of its weight.

2,3: pipes corresponding to edges are introduced at an altitude \( \eta_{pq} = \nu_p \lor \nu_q = (\delta_{en} \nu)_{pq} \)

4: the tank network corresponding to these edge weights

\( \eta_{pq} = \nu_p \lor \nu_q = (\delta_{en} \nu)_{pq} \)
5: node critical filling of the tanks (4). The node weights are
\[ \nu = \varepsilon_{ne}\delta_{en}\nu = \varphi_n\nu. \]

6: the new level of water in the tank C is now \( \nu = \varepsilon_{ne}\delta_{en}\nu = \varphi_n\nu > \nu \): the closing \( \varphi_n \) increases the level of the isolated regional minima (such as C), leaving unchanged the level in all other tanks.
1: tank network associated to a node weighted graph.
2: pipes are introduced and to produce an edge critical filling of the tanks. An "auto-pipe" connecting the tank \( c \) with itself is added, at the level of its filling. The level of water in the tanks is now invariant by the closing \( \varphi_n : \nu = \varphi_n \nu \).

The weights of the pipes are 
\[
\eta_{pq} = \nu_p \vee \nu_q = (\delta_{en} \nu)_{pq} = (\delta_{en} \varphi_n \nu)_{pq} = (\delta_{en} \varepsilon_{ne} \delta_{en} \nu)_{pq} = (\gamma_e \delta_{en} \nu)_{pq} = (\gamma_e \eta)_{pq}
\]

And \( \nu = \varphi_n \nu = \varepsilon_{ne} \delta_{en} \nu = \varepsilon_{ne} \eta \). The filling of the tanks is node and edge critical.
1: Tank network of an edge weighted graph with pipes at altitude $\eta$.
2,3: The tanks are filled at level $\varepsilon_{ne}\eta$
4: Tanks, without pipes with this level $\varepsilon_{ne}\eta$
5: new pipes at level $\eta' = \delta_{en}\varepsilon_{ne}\eta = \gamma_e\eta$
6: The pipes verifying $\gamma_e\eta < \eta$ are those which are not the lowest of one of their extremities
Critical fillings of the tank network of an edge weighted graph

1: tank network associated to an edge weighted graph
2: By suppressing all edges of the which are not the lowest edges of one of their neighbors (verifying $\eta > \gamma_e \eta$), a new tank network is produced with altitudes $\eta = \gamma_e \eta = (\delta_{en} \epsilon_{ne} \eta)$.
3: a node critical filling at level $\nu = \epsilon_{ne} \eta = (\epsilon_{ne} \gamma_e \eta) = (\epsilon_{ne} \delta_{en} \epsilon_{ne} \eta) = (\varphi_n \epsilon_{ne} \eta) = (\varphi_n \nu)$.

As $\eta = \gamma_e \eta = (\delta_{en} \epsilon_{ne} \eta)$, we have $\eta = \gamma_e \eta = \delta_{en} \nu$.

The filling of the tanks is node and edge critical.
The flowing adjunction
The flowing adjunction

**Theorem**

The pair of operators $(\delta_{en}, \varepsilon_{ne})$ form an adjunction: \( \forall \eta \in F_e, \forall \nu \in F_n : \delta_{en} \nu \leq \eta \iff \nu \leq \varepsilon_{ne} \eta. \)

**Proof.**

Consider for the same graph an arbitrary function \( \nu \in F_n \) on the nodes and an arbitrary function \( \eta \in F_e \) on the edges. Then

\[
\delta_{en} \nu \leq \eta \iff \forall i, j : \nu_i \lor \nu_j \leq \eta_{ij} \iff \forall i, j : \nu_i \leq \eta_{ij} \iff \forall i, j : \nu_i \leq \varepsilon_{ne} \eta_{ij}
\]

\((j \text{ neighbors of } i)\)
The flowing adjunction

It results:

- $\delta_{en}$ is a dilation from $\mathcal{F}_n$ into $\mathcal{F}_e$
- $\varepsilon_{ne}$ is an erosion from $\mathcal{F}_e$ into $\mathcal{F}_n$
- $\gamma_e = \delta_{en}\varepsilon_{ne}$ is an opening from $\mathcal{F}_e$ into $\mathcal{F}_e$
- $\varphi_n = \varepsilon_{ne}\delta_{en}$ is a closing from $\mathcal{F}_n$ into $\mathcal{F}_n$
The flowing edges

We call "n-flowing" edge a flowing edge of a node weighted graph and e-flowing edge a flowing edge of an edge weighted graph.

Lemma

"n-flowing": Each n-flowing edge in the graph $G(\nu, \text{nil})$ also is an e-flowing edge in the graph $G(\text{nil}, \delta_{en}\nu)$

Proof.

An edge $e_{pq}$ is an n-flowing edge of $p$ iff $\nu_p \geq \nu_q$. Thus the weight assigned to $e_{pq}$ is $(\delta_{en}\nu)_{pq} = \nu_p \lor \nu_q = \nu_p$. For each other neighbor $s$ of $p$, we have $(\delta_{en}\nu)_{ps} = \nu_p \lor \nu_s \geq \nu_p$. Hence $e_{pq}$ is one of the edges with minimal weight adjacent to $p$, i.e. an e-flowing edge of $G(\text{nil}, \delta_{en}\nu)$. \qed
The flowing edges

Lemma

"e-flowing": Each e-flowing edge in the graph $G(nil, \eta)$ also is an n-flowing edge in the graph $G(nil, \epsilon_ne\eta)$

Proof.

Consider an e-flowing edge $e_{pq}$ of $G(nil, \eta)$, i.e. $e_{pq}$ is one of the adjacent edges of $p$ with minimal weight. Then $\nu_p = (\epsilon_ne\eta)_p = \eta_{pq}$. And $\nu_q = (\epsilon_ne\eta)_q \leq \eta_{pq}$, as $e_{pq}$ is one of the adjacent edges of $q$, but not necessarily the smallest of them. Hence $\nu_p \geq \nu_q$ and $e_{pq}$ indeed also is an n-flowing edge of $G(nil, \epsilon_ne\eta)$.
Applying both lemmas in sequence:
\[ \{e_{pq} \text{ n-flowing edge of } G(\nu, \text{nil})\} \subset \{e_{pq} \text{ e-flowing edge of } G(\text{nil}, \delta_{en}\nu)\} \subset \{e_{pq} \text{ n-flowing edge of } G(\text{nil}, \varepsilon_{ne}\delta_{en}\nu) = G(\text{nil}, \varphi_n\nu)\} \]

**Corollary**

If the function \( \nu \) is closed by \( \varphi_n : \nu = \varphi_n\nu = \varepsilon_{ne}\delta_{en}\nu \), then the n-flowing edges of \( G(\nu, \text{nil}) \) are identical with the e-flowing edge of \( G(\text{nil}, \delta_{en}\nu) \).

**Corollary**

\( \nu = \varphi_n\nu : G(\nu, \text{nil}) \) and \( G(\text{nil}, \delta_{en}\nu) \) having the same flowing edges, are associated to the same gravitational graph: they also have the same flowing paths, regional minima and catchment zones.
The flowing edges

Applying both lemmas in sequence:
\[ \{e_{pq} \text{ e-flowing edge of } G(nil, \eta)\} \subseteq \{e_{pq} \text{ n-flowing edge of } G(\varepsilon_{ne}\eta, nil)\} \subseteq \{e_{pq} \text{ e-flowing edge of } G(nil, \delta_{en}\varepsilon_{ne}\eta)\} \]

**Corollary**

If the function \( \eta \) is open by \( \gamma_e : \eta = \gamma_e\eta = \delta_{en}\varepsilon_{ne}\eta \), then the e-flowing edges of \( G(nil, \eta) \) are identical with the n-flowing edge of \( G(\varepsilon_{ne}\eta, nil) \)

**Corollary**

\( \eta = \gamma_e\eta : G(nil, \eta) \text{ and } G(\varepsilon_{ne}\eta, nil) \) having the same flowing edges, are associated to the same gravitational graph: they also have the same flowing paths, regional minima and catchment zones.
the invariance domain of the opening and the closing
The invariance domain of the closing

Consider in a graph $G(nil, \nu)$ a node $p$ with a weight $\lambda$. For each neighboring node $s$ of $p$: $(\delta_{en}\nu)_{ps} = \nu_p \lor \nu_s \geq \lambda$. Two possibilities exist for $p$:

- $p$ is an isolated regional minimum. For each neighboring node $q$, $\nu_q > \lambda$, and $(\delta_{en}\nu)_{pq} > \lambda$. The subsequent erosion $\varepsilon_{ne}$ assigns to $p$ the smallest of these weights, also higher than $\lambda$. Hence if $p$ is an isolated regional minimum, its weight is increased by the closing $\varphi_n$.

- the node $p$ is not an isolated regional minimum and has a neighbor $q$ with a weight $\mu \leq \lambda$ and $(\delta_{en}\nu)_{pq} = \lambda$. As for each other neighbor of $p$, $(\delta_{en}\nu)_{ps} \geq \lambda$, the subsequent erosion $\varepsilon_{ne}$ assigns to $p$ the smallest of these weights, that is $\lambda$. Hence, if $p$ is not an isolated regional minimum, its weight is invariant by the closing $\varphi_n$.

Lemma

The graph $G(nil, \nu)$ is invariant by the $\varphi_n$ if and only if it has no isolated regional minima.
Editing a node weighted graph to make it invariant by the closing

Adding an edge linking $p$ with itself creates a loop edge, with a weight $\delta_{en} \nu_p$. With the subsequent erosion $\varepsilon_{ne}$, we have $\varepsilon_{en} \delta_{en} \nu_p = \nu_p$.

**Corollary**

Adding to each isolated regional minimum a loop edge transforms the graph $G(nil, \nu)$ into a graph $\odot G(nil, \nu)$ where $\nu$ is invariant by $\varphi_n$. Both graphs have the same regional minima and catchment zones.

Adding self-loop edges to the regional minima does not change the flowing paths outside the minima and the regional minima remain regional minima with the same nodes. Hence the new graph $\odot G(nil, \nu)$ has the same catchment zones as the initial graph.

**Theorem**

The graph $\odot G(\delta_{en} \nu, \nu)$ is a flowing graph. If $\eta = \delta_{en} \nu$, its node weights verify $\nu = \varphi_n \nu = \varepsilon_{ne} \delta_{en} \nu = \varepsilon_{ne} \eta$. 

Fernand Meyer (Centre de Morphologie MathA hydostatic model representing edge or node)
Editing a node weighted graph to make it invariant by the closing

A hydostatic model representing edge or node weighted graphs.
Editing a node weighted graph to make it invariant by the closing
The invariance domain of the opening

Consider a graph $G(\eta, \text{nil})$. Two possibilities exist for an edge $e_{pq}$ with a weight $\eta_{pq} = \lambda$:

* $e_{pq}$ is not a flowing edge, i.e. is not the lowest neighboring edge of one of its extremities: it has lower neighboring edges at each extremity. Hence $(\varepsilon_{ne}\eta)_p < \lambda$ and $(\varepsilon_{ne}\eta)_q < \lambda$; hence $(\gamma_e\eta)_{pq} = (\delta_{en}\varepsilon_{ne}\eta)_{pq} = (\varepsilon_{ne}\eta)_p \lor (\varepsilon_{ne}\eta)_q < \lambda$. Each non flowing edge is lowered by the opening $\gamma_e$.

* the edge $e_{pq}$ is a flowing edge and is the lowest edge of one of its the extremities, say $p$. Then $(\varepsilon_{ne}\eta)_p = \lambda$ and $(\varepsilon_{ne}\eta)_q \leq \lambda$; hence $(\delta_{en}\varepsilon_{ne}\eta)_{pq} = (\varepsilon_{ne}\eta)_p \lor (\varepsilon_{ne}\eta)_q = \lambda$. Each flowing edge is invariant by the opening $\gamma_e$.

Lemma

The graph $G(\eta, \text{nil})$ is invariant by the opening $\gamma_e$ if and only if all its edges are flooding edges.
Editing an edge weighted graph to make it invariant by the opening

The operator $\varepsilon_{ne}\eta$ assigns to each node the weight of its adjacent flowing edges. Hence suppressing the non flowing edges in the graph $G(nil, \eta)$ does not modify the erosion $\varepsilon_{ne}\eta$, that is the weight of the nodes.

**Lemma**

Suppressing all edges of the graph $G(nil, \eta)$ whose weight is lowered by the opening $\gamma_e$ produces a graph where each edge is a flowing edge with a weight invariant by $\gamma_e$. Both graphs have the same regional minima and catchment zones.

As shown above, the non flowing edges are lowered by the opening $\gamma_e$, whereas the flowing edges are not. Suppressing the edges verifying $\gamma_e\eta < \eta$ leaves a pruned graph on which $\gamma_e\eta = \eta$. 
Suppressing the edges lowered by $\gamma_e$ leaves only the flowing edges. As a drop of water never follows a non-flowing edge, the flowing paths are not modified if such edges are cut. The regional minima are not modified either as their inside edges are all flowing edges. Hence the new graph $\downarrow G(nil, \eta)$ has the same catchment zones as the initial graph.

**Theorem**

The graph $\downarrow G(\varepsilon_{ne} \eta, \eta,)$ is a flowing graph. If $\nu = \varepsilon_{ne} \eta$, we have $\eta = \gamma_e \eta = \delta_{en} \varepsilon_{ne} \eta = \delta_{en} \nu$. 
Without editing the graph: no correspondance between the minima

A: an edge weighted graph with edge weights $\eta$; the regional minima have red edges

B: $\nu = \varepsilon_{ne}\eta$; the node based regional minima are different as the edge based regional minima

C: $\gamma_e\eta = \delta_{en}\varepsilon_{ne}\eta$: the edges verifying $\eta > \gamma_e\eta$ are non flowing edges; the edge based regional minima are different in A and C.
Editing an edge weighted graph to make it invariant by the opening

D: the non flowing edges have been suppressed and the edge weights verify \( \eta = \gamma_e \eta \) on this new graph

E: the node weights \( \varepsilon_{ne} \eta \) obtained on the pruned graph are unchanged. The flowing edges remain the same, thus the regional minima and catchment zones are identical in A, E and F.
Properties of a flowing graph
Definition

A node and edge weighted graph $G(\nu, \eta)$ is a flowing graph if $\nu$ and $\eta$ verify the double coupling: $\nu = \varepsilon_{ne} \eta$ and $\eta = \delta_{en} \nu$.

Replacing $\nu$ by its value we get $\eta = \delta_{en} \nu = \delta_{en} \varepsilon_{ne} \eta = \gamma_e \eta$. Replacing $\eta$ by its value we get $\nu = \varepsilon_{ne} \eta = \varepsilon_{ne} \delta_{en} \nu = \varphi_n \nu$. Hence:

Corollary

A flowing graph is invariant by the opening $\gamma_e$ and by the closing $\varphi_n$. 
The edges of a flowing graph

Consider an edge $e_{pq}$. As $\eta_{pq} = (\delta_{en} \nu)_{pq} = \nu_p \lor \nu_q$, its weight is equal to the weight of one of its extremities, say $p$, and $\eta_{pq} = \nu_p \geq \nu_q$. The edge $e_{pq}$ is thus a n-flooding edge of $G(\nu, nil)$. On the other hand, as $\nu_p = (\varepsilon_{ne} \eta)_p = \land \eta_{pr}$, the weight $\eta_{pq} = \nu_p$ is the smallest weight of the edges adjacent to $p$: the edge $e_{pq}$ is also a e-flooding edge of $G(nil, \eta)$. Each node is the extremity of a flowing edge with the same weight.

**Lemma**

Each edge in a flooding graph $G(\eta, \nu)$ is a n-flooding edge in the graph $G(nil, \nu)$ and an e-flooding edge in the graph $G(\eta, nil)$.

**Corollary**

Having the same flowing edges, $G(\eta, \nu)$ and $G(nil, \nu)$ have the same gravitational graph, flowing paths, regional minima and catchment zones.
Transforming a node weighted graph into a flowing graph
Transforming a node weighted graph into a flowing graph

Starting with a node weighted graph $G(v, \text{nil})$, we add self-loop edges to the regional minima. On this new graph $G(v, \text{nil})$, the node weights verify $v = \varphi_n v = \epsilon_{ne} \delta_{en} v$.

**Theorem**

The graph $G(v, \delta_{en} v)$ is a flowing graph. If $\eta = \delta_{en} v$, its node weights verify $v = \varphi_n v = \epsilon_{ne} \delta_{en} v = \epsilon_{ne} \eta$. 
From $G(nil, \nu)$ to $G(nil, \delta_{en} \nu)$: not the same flowing paths and regional minima.
From \( G(\nu, nil) \) to \( \otimes G(\nu, nil) \) : the same flowing paths and regional minima.
From \( G(v, \text{nil}) \) to \( G(v, \delta_en\nu) \): the same flowing paths and regional minima.

\[ \text{Graph B} \quad \leftrightarrow \quad \text{Graph C} \]
From $\mathcal{G}(\nu, \delta_{en}\nu)$ to $\mathcal{G}(\text{nil}, \delta_{en}\nu)$: the same flowing paths and regional minima.
From $G(\nu, \delta_{en}\nu)$ to $G(nil, \delta_{en}\nu)$: the same flowing paths and regional minima.
From $\overset{\circ}{\nabla} G(nil, \delta_{en}v)$ to $\overset{\circ}{\nabla} G(nil, nil)$: the same flowing paths and regional minima.
From $G(\nu, \text{nil})$ to $\bigcirc G(\text{nil}, \text{nil})$ : the same flowing paths and regional minima.
Transforming an edge weighted graph into a flowing graph
Transforming an edge weighted graph into a flowing graph

Suppressing all edges of the graph \( G(nil, \eta) \) whose weight is lowered by the opening \( \gamma_e \) produces a graph \( \downarrow G(nil, \eta) \) in which \( \eta = \delta_{en} \varepsilon_{ne} \eta = \gamma_e \eta \).

Theorem

The graph \( \downarrow G(\varepsilon_{ne} \eta, \eta) \) is a flowing graph. If \( \nu = \varepsilon_{ne} \eta \), we have \( \eta = \gamma_e \eta = \delta_{en} \varepsilon_{ne} \eta = \delta_{en} \nu \).
From $G(nil, \eta)$ to $G(\varepsilon_{ne}\eta, nil)$: not the same flowing paths and regional minima.
From $G(nil, \eta)$ to $\downarrow G(nil, \eta)$: the same flowing paths and regional minima.
From $G(nil, \eta)$ to $G(\varepsilon_{ne}\eta, \eta)$: the same flowing paths and regional minima.
From $\downarrow G(\varepsilon_{ne}\eta, \eta)$ to $\downarrow G(\varepsilon_{ne}\eta, nil)$: the same flowing paths and regional minima.
From $\downarrow G(\varepsilon_{ne}\eta, \eta)$ to $\overrightarrow{G}(\varepsilon_{ne}\eta, \eta)$: the same flowing paths and regional minima.
From $\vec{G}(\varepsilon_{\text{ne}}, \eta, \eta)$ to $\vec{G}(\text{nil}, \text{nil})$: the same flowing paths and regional minima.
From $G(nil, \eta)$ to $\overrightarrow{G}(nil, nil)$: the same flowing paths and regional minima.