Filters and levelings

Centre for Mathematical Morphology
Table of contents

1. Morphological filters
2. Composing morphological filters
3. Floodings and razings
4. Levelings
Morphological filters
Definition (Morphological filter - G. Matheron, J. Serra)

A **morphological filter** $\phi$ is an **increasing** and **idempotent** transformation from a complete lattice $\mathcal{L}$ into itself:

- $\phi : \mathcal{L} \rightarrow \mathcal{L}$
- $\forall x, y \in \mathcal{L}, x \leq y \Rightarrow \phi(x) \leq \phi(y)$
- $\phi \circ \phi = \phi$

**Reminder:** A complete lattice is a partially ordered set for which any subset admits an infimum and a supremum.

**Remark:** A non-trivial morphological filter ($\phi \neq id$) is not invertible. Therefore, the filtering process implies a loss of information.
Do you know morphological filters already?
Examples of filters

- **Openings and closings**
  - An opening $\gamma$ is an anti-extensive filter: $\gamma \leq id$
  - $\delta \varepsilon$ where $(\varepsilon, \delta)$ is an adjunction
  - Openings by reconstruction (see course on geodesy)
  - Area opening
  
  A closing $\varphi$ is an anti-extensive filter: $id \leq \varphi$
  - $\varepsilon \delta$ where $(\varepsilon, \delta)$ is an adjunction
  - Closings by reconstruction

- **Some composition of filters**
  We will see that under certain conditions, composing filters produce new filters.
Opening by a hexagonal structuring element.
Opening by reconstruction using the previous openings.
Opening by a hexagonal structuring element.
Opening by reconstruction using the previous openings.
Area opening, with the connectivity defined by a hexagonal structuring element.
Area opening, with the connectivity defined by a hexagonal structuring element.
Closing by a hexagonal structuring element.
Closing by reconstruction using the previous closings.
Closing by a hexagonal structuring element.
Closing by reconstruction using the previous closings.
Overfilters, Underfilters

Definition (Overfilter)

An overfilter $\phi$ is an increasing and overpotent transformation from a complete lattice $\mathcal{L}$ into itself:

- $\phi : \mathcal{L} \to \mathcal{L}$
- $\forall x, y \in \mathcal{L}, x \leq y \Rightarrow \phi(x) \leq \phi(y)$
- $\phi \circ \phi \geq \phi$

Definition (Underfilter)

An underfilter $\psi$ is an increasing and underpotent transformation from a complete lattice $\mathcal{L}$ into itself:

- $\psi : \mathcal{L} \to \mathcal{L}$
- $\forall x, y \in \mathcal{L}, x \leq y \Rightarrow \psi(x) \leq \psi(y)$
- $\psi \circ \psi \leq \psi$
Examples of overfilters and underfilters

Let $B$ be a flat structuring element which includes the origin. Then

- the dilation $\delta_B$ is an overfilter;
- the erosion $\varepsilon_B$ is an underfilter.

Let $I$ be a set and $(\psi)_{i \in I}$ a family of filters of $\mathcal{L}$. Then

- $\bigvee_{i \in I} \psi_i$ is an overfilter;
- $\bigwedge_{i \in I} \psi_i$ is an underfilter.

Morphological filters are overfilters and underfilters.
Sup-filters, Inf-filters and strong filters

Definition (Sup-filter, inf-filter, strong filter)

Let $\psi : \mathcal{L} \to \mathcal{L}$ be a filter from a complete lattice $\mathcal{L}$ into itself. Then $\psi$ is called

- a sup-filter if $\psi = \psi(id \lor \psi)$;
- an inf-filter if $\psi = \psi(id \land \psi)$;
- a strong-filter if $\psi = \psi(id \lor \psi) = \psi(id \land \psi)$.

Interpretation: these notions characterize the robustness of the operator $\psi$ to perturbations on the input signal. Indeed, for $f, g \in \mathcal{L}$,

- if $\psi$ is a sup-filter, then $f \leq g \leq f \lor \psi(f) \Rightarrow \psi(f) = \psi(g)$;
- if $\psi$ is an inf-filter, then $f \land \psi(f) \leq g \leq f \Rightarrow \psi(f) = \psi(g)$;
- if $\psi$ is a strong filter, then $f \land \psi(f) \leq g \leq f \lor \psi(f) \Rightarrow \psi(f) = \psi(g)$.

Examples: Openings and closings are strong filters.
Sup-filters, Inf-filters and strong filters

Definition (Sup-filter, inf-filter, strong filter)

Let \( \psi : \mathcal{L} \rightarrow \mathcal{L} \) be a filter from a complete lattice \( \mathcal{L} \) into itself. Then \( \psi \) is called

- a sup-filter if \( \psi = \psi(id \lor \psi) \);
- an inf-filter if \( \psi = \psi(id \land \psi) \);
- a strong-filter if \( \psi = \psi(id \lor \psi) = \psi(id \land \psi) \).

Image credit: Jean-Serra, Course on Mathematical Morphology, First Part, 1999.
Theorem (G. Matheron)

- A mapping $\psi : \mathcal{L} \to \mathcal{L}$ is a sup-filter (resp. inf-filter) if and only if there exist an opening $\gamma$ and a closing $\varphi$ such that $\psi = \gamma \circ \varphi$ (resp. $\psi = \varphi \circ \gamma$).
- If $\gamma$ is an opening and $\varphi$ a closing, then
  - $\varphi$ and $\gamma$ are strong filters,
  - $\gamma \varphi$ and $\varphi \gamma \varphi$ are sup-filters,
  - $\varphi \gamma$ and $\gamma \varphi \gamma$ are inf-filters.
Composing morphological filters
The theorem we just saw implies that if $\gamma$ is an opening and $\varphi$ a closing, then the four compositions $\varphi \gamma$, $\gamma \varphi$, $\varphi \gamma \gamma$, $\gamma \varphi \gamma$ are also filters.

Can we generalize this to any pair of filters $(\xi, \psi)$?

The following result states we can, as soon as they are ordered.
Alternating filters

Theorem (G. Matheron [Serra, 1988, Serra and Vincent, 1992])
Let $\psi$ and $\xi$ be two ordered filters (suppose $\xi \geq \psi$). Then

1. $\xi\psi, \psi\xi, \psi\xi\psi$ and $\xi\psi\xi$ are morphological filters
2. $\psi \leq \psi\xi\psi \leq \xi \land \psi\xi \leq \psi\psi \lor \psi\xi \leq \xi\psi\xi \leq \xi$
3. $\xi\psi\xi$ is the smallest filter greater than $\xi\psi \lor \psi\xi$ and $\psi\xi\psi$ is the greatest filter smaller than $\xi\psi \land \psi\xi$
4. The following equivalences hold:
   \[
   \xi\psi\xi = \psi\xi \iff \psi\xi\psi = \xi\psi \iff \psi\xi \geq \xi\psi.
   \]

Remarks:

- The composition of more than three operators among $\{\psi, \xi\}$ yields no new filter.
- This result generalizes to $\psi$ overfilter, $\xi$ underfilter and $\xi \geq \psi$ [Heijmans, 1997].
A very important generalization concerns families of ordered underfilters and overfilters.
Alternating sequential filters

Theorem ([Heijmans, 1997, Bloch et al., 2007])

Let \((\xi_i)_{i \in \mathbb{N}^*}\) an increasing family of underfilters and \((\psi_i)_{i \in \mathbb{N}^*}\) a decreasing family of overfilters, such that \(\psi_1 \leq \xi_1\), that is to say:

\[
\cdots \leq \psi_n \leq \cdots \leq \psi_2 \leq \psi_1 \leq \xi_1 \leq \xi_2 \leq \cdots \leq \xi_n \leq \cdots
\]

Then any composition of these operators, containing at least one \(\psi_i\) and one \(\xi_i\), is a filter.

A consequence of this theorem is the following proposition.

Proposition (Adjunctional filters [Heijmans, 1997, Bloch et al., 2007])

Let \((\varepsilon, \delta)\) be an adjunction on \(\mathcal{L}\). Then any repeated composition of \(\varepsilon\) and \(\delta\) in any order, containing the same number of instances of \(\varepsilon\) and \(\delta\), is a filter, called a adjunctional filter.

For example, \(\psi = \delta \varepsilon^3 \delta^4 \varepsilon^2\) is a filter and so is \(\psi = \delta^4 \varepsilon^2 \delta \varepsilon^3\).
Alternating sequential filters

A less general version of the previous theorem was proposed by Jean Serra [Serra, 1988, Serra and Vincent, 1992] and concerned a more specific class of compositions, still widely used in practice, especially on noisy signals.

Definition (Alternating sequential filters)

Let \((\xi_i)_{i \in \mathbb{N}^*}\) an increasing family of underfilters and \((\psi_i)_{i \in \mathbb{N}^*}\) a decreasing family of overfilters, such that \(\psi_1 \leq \xi_1\).

Then for any \(i \geq 1\), the filters

\[ M_i = \psi_i \xi_i \ldots \psi_2 \xi_2 \psi_1 \xi_1 \quad N_i = \xi_i \psi_i \ldots \xi_2 \psi_2 \xi_1 \psi_1 \]

are called Alternating sequential filters.
Alternating sequential filters

In this example a simple $\varphi \gamma$ is not appropriate to cancel noise, whatever the size of the structuring element. An alternating sequential filter allows noise cancellation and smoothly produces a good approximation of the signal without noise.

Images and text credit: Jean-Serra, Course on Mathematical Morphology, First Part, 1999.
Compositions of closings $\varphi_i = \varepsilon^i_B \delta^i_B$ and openings $\gamma_i = \delta^i_B \varepsilon^i_B$, where $B$ is hexagonal.
Compositions of closings $\varphi_i = \varepsilon^i_B \delta^i_B$ and openings $\gamma_i = \delta^i_B \varepsilon^i_B$, where $B$ is hexagonal.
Compositions of a closing by reconstruction $\varphi_i = rec_{ero}(\varepsilon_i^B \delta_B^i(\cdot), \cdot)$ and an opening by reconstruction $\gamma_i = rec_{dil}(\delta_B^i \varepsilon_i^B(\cdot), \cdot)$, where $B$ is hexagonal.
Compositions of a closing by reconstruction $\varphi_i = \text{rec}_\text{ero}(\varepsilon_B^i \delta_B^i(\cdot), \cdot)$ and an opening by reconstruction $\gamma_i = \text{rec}_\text{dil}(\delta_B^i \varepsilon_B^i(\cdot), \cdot)$, where $B$ is hexagonal.
Evolution of the regional minima.
Evolution of the regional maxima.
Alternating sequential filters of the type $\gamma_i \varphi_i \ldots \gamma_1 \varphi_1$, where $\varphi_i = \text{rec}_\text{ero}(\varepsilon_i^1 \delta_i^1(\cdot), \cdot)$ is a closing by reconstruction, $\gamma_i = \text{rec}_\text{dil}(\delta_i^1 \varepsilon_i^1(\cdot), \cdot)$ an opening by reconstruction, and $B$ is hexagonal.
Alternating sequential filters of the type φ_1γ_1 \ldots φ_1γ_1, where φ_i = rec_{ero}(ε_i^B δ_i^B(\cdot), \cdot) is a closing by reconstruction, γ_i = rec_{dil}(δ_i^B ε_i^B(\cdot), \cdot) an opening by reconstruction, and B is hexagonal.
Evolution of the regional maxima for the alternating sequential filters of the type $\gamma_i \phi_i \cdots \gamma_1 \phi_1$. 
Evolution of the regional minima for the alternating sequential filters of the type \( \gamma \psi \cdot \cdot \cdot \gamma \psi \).
Floodings and razings
A flooding of a function $f$ is simply a geodesic reconstruction by erosion from a marker function $h$ (closing by reconstruction).

Dually, a **razing** of $f$ is simply a geodesic reconstruction by dilation from a marker function $h$ (opening by reconstruction).

Idea

Flooding and razing are convenient terms, as shorter and illustrative of the properties of the corresponding transformations. Also, floodings and razings may be studied as sets of functions associated to $f$, regardless of the markers. Leaving aside the word “reconstruction” is consistent with such a focus.

Let $\mathcal{L}$ be the complete lattice of images of a fixed size, $f \in \mathcal{L}$ one such image and $B$ a symmetric structuring element containing the origin.

**Definition (Geodesic reconstruction by dilation)**

The reconstruction by dilation of $f$ is the operator defined for any marker function $h \in \mathcal{L}$ by

$$R_f(h) = \lim_{n \to +\infty} g_n = \bigvee_{n \geq 0} g_n$$

where the sequence $(g_n) \in \mathcal{L}^\mathbb{N}$ is recursively defined by

\[
\begin{cases}
  g_0 = f \wedge h \\
  g_{n+1} = \delta_B(g_n) \wedge f.
\end{cases}
\]

For a fixed marker $h$, the operator $\psi : f \in \mathcal{L} \mapsto R_f(h)$ is an opening on $\mathcal{L}$, also called opening by reconstruction from the marker $h$. 
Definition (Geodesic reconstruction by erosion)

The reconstruction by erosion of $f$ is the operator defined for any marker function $h \in \mathcal{L}$ by

$$R_f^*(h) = \lim_{n \to +\infty} g_n = \bigwedge_{n \geq 0} g_n$$

where the sequence $(g_n) \in \mathcal{L}^\mathbb{N}$ is recursively defined by

$$
\begin{cases}
  g_0 & = f \lor h \\
  g_{n+1} & = \varepsilon_B(g_n) \lor f.
\end{cases}
$$

For a fixed marker $h$, the operator $\xi : f \in \mathcal{L} \mapsto R_f^*(h)$ is a closing on $\mathcal{L}$, also called closing by reconstruction from the marker $h$.

Since $B$ is symmetrical, $\varepsilon_B$ is the dual of $\delta_B$ and we have $R_f^*(h) = (R_f(h^c))^c$. 
Definition (Flooding)

We say that $g \in \mathcal{L}$ is a flooding of $f \in \mathcal{L}$ if and only if there exists $h \in \mathcal{L}$ such that

$$g = R_f^*(h).$$
Floodings

Characterization:
We have the three following characterizations of floodings:

• $g$ is a flooding of $f$ if and only if $g \geq f$ and for any couple of neighbouring pixels $(p, q)$:

$$g_p > g_q \Rightarrow g_p = f_p.$$ 

• $g$ is a flooding of $f$ if and only if $g \geq f$ and for any couple of neighbouring pixels $(p, q)$:

$$f_p \leq g_p \leq f_p \lor g_q$$ 

• $g$ is a flooding of $f$ if and only if $g = f \lor \varepsilon_B(g)$

Floodings starting with openings of different sizes $i$ (hexagonal structuring element).
Evolution of the regional maxima for floodings starting with openings of different sizes.
Evolution of the regional minima for floodings starting with openings of different sizes.
Floodings

Properties:

• The infimum of floodings of $f$ is also a flooding of $f$

• **floodings do not create new regional extrema.** Let $g$ be a flooding of $f$, then:
  • If the set of pixels $X$ is a regional maximum of $g$, then $X$ is also a regional maximum of $f$.
  • If $X$ is a regional minimum of $g$, then $X$ contains a regional minimum of $f$.

• Floodings are strong filters.
Definition (Razing)

We say that $g \in \mathcal{L}$ is a razing of $f \in \mathcal{L}$ if and only if there exists $h \in \mathcal{L}$ such that

$$g = R_f(h).$$

Razings

Characterization:

We have the three following characterizations of razings:

• $g$ is a razing of $f$ if and only if $g \preceq f$ and for any couple of neighbouring pixels $(p, q)$:
  \[ g_p > g_q \Rightarrow g_q = f_q. \]

• $g$ is a razing of $f$ if and only if $g \succeq f$ and for any couple of neighbouring pixels $(p, q)$:
  \[ f_q \land g_p \leq g_q \leq f_q \]

• $g$ is a razing of $f$ if and only if $g = f \land \delta_B(g)$

Razings starting with closings of different sizes $i$ (hexagonal structuring element).
Evolution of the regional maxima for razings starting with closings of different sizes.
Evolution of the regional minima for razings starting with closings of different sizes.
Razings

Properties:

• The supremum of razings of $f$ is also a razing of $f$

• **Razings do not create new regional extrema.** Let $g$ be a razing of $f$, then:
  • If the set of pixels $X$ is a regional minimum of $g$, then $X$ is also a regional minimum of $f$.
  • If $X$ is a regional maximum of $g$, then $X$ contains a regional maximum of $f$.

• Razings are strong filters.
Levelings
Let $f, g, h \in \mathcal{L}$. We denote by $Fl_h(.)$ the flooding $R^*(h)$ and by $Rz_h(.)$ the razing $R_z(h)$. These two filters commute, yielding a new filter:

**Definition (Leveling)**

We say that $g$ is a **leveling of $f$** if and only if there exists $h \in \mathcal{L}$ such that

$$g = Fl_h \circ Rz_h(f) = Rz_h \circ Fl_h(f).$$

Levelings

Characterization:
We have the four following characterizations of levelings. g is a leveling of f:

- if and only if for any couple of neighbouring pixels \((p, q)\): \(g_q < g_p \Rightarrow f_q \leq g_q \) and \(g_p \leq f_p\).
- if and only if for any couple of neighbouring pixels \((p, q)\): \(f_q \land g_p \leq g_q \leq f_q \lor g_p\)
- if and only if \(f \land \delta_B(g) \leq g \leq f \lor \varepsilon_B(g)\)
- if and only if \(g = (f \land \delta_B(g)) \lor \varepsilon_B(g) = (f \lor \varepsilon_B(g)) \land \delta_B(g)\)

Levelings

Properties:

• The supremum and infimum of levelings of $f$ is also a leveling of $f$

• Levelings do not create new regional minima or maxima : each regional minimum (resp. maximum) of the leveling contains a regional minimum (resp. maximum) of the initial function

• Floodings and razings are particular levelings

• Levelings are strong, auto-dual filters.

• The leveling $g$ of $f$ with respect to $h$ can also be constructed as the limit of the sequence $(g)_{n \in \mathbb{N}}$ defined by

$$
\begin{align*}
  g_0 &= h \\
  g_{n+1} &= (f \wedge \delta_B(g_n)) \lor \varepsilon_B(g_n) = (f \lor \varepsilon_B(g_n)) \land \delta_B(g_n)
\end{align*}
$$
Levelings using as markers alternating sequential filters of increasing sizes $i$ (hexagonal structuring element).
Evolution of the regional maxima for levelings using as markers alternating sequential filters of increasing sizes.
Evolution of the regional minima for levelings using as markers alternating sequential filters of increasing sizes.
Original

\[ i = 1 \quad i = 3 \quad i = 5 \quad i = 7 \]

Levelings using as markers alternating sequential filters of increasing sizes \( i \) (hexagonal structuring element).
Evolution of the regional maxima for levelings using as markers alternating sequential filters of increasing sizes.
Evolution of the regional minima for levelings using as markers alternating sequential filters of increasing sizes.
Lambda levelings

Quasi (or $\lambda$)-flat zones instead of flat zones.
In this transformation, the dilation and erosion are replaced by “viscous” counterparts:

\[ g_{n+1} = (f \lor [g_n \land \varepsilon_B(\varphi(g_n))]) \land (g_n \lor \delta_B(\gamma(g_n))), \]

where \( \varphi \) and \( \gamma \) are for example a closing and an opening.

