Introduction to stochastic geometry

Bruno Figliuzzi

February 2, 2015

ii

Contents

II Appendix

Chapter 1

Introduction

Most materials used in contemporary life and industry are heterogeneous and exhibit a complex internal microstructure. The microstructure is a key feature of the global material, which largely determines most of its physical properties at the macroscopic level [35].

From an experimental perspective, one of the most efficient way to study microstructures of heterogenenous media is to rely on images obtained with experimental techniques including tomography or microscopy. Images provide a large amount of information on the studied microstructures and therefore require sophisticated image processing and statistical tools to be processed. The first step of the processing aims at developing segmentation algorithms to properly identify the micro-structure components. Measurements can next be defined and applied on the segmented images. Thus, starting from experimental images, one obtains a description of the studied microstructure through geometrical features that aggregate its main geometrical characteristics.

Conversely, one is often interested in generating random microstructures that reproduce accurately some geometrical features of the original material. The simulated microstructures can in turn serve as a basis to investigate the physical or mechanical properties of heterogeneous materials through extensive numerical simulations. This approach is of interest to better understand the influence of the microstructure on the physical properties of the material at the macroscopic scale.

Mathematical morphology and stochastic geometry provide efficient tools for both analysis and simulation of heterogeneous microstructures. In this introduction, our goal is to provide a brief overview of these research fields.

1.1   Experimental image segmentation and mathematical morphology

When exploiting experimental images of heterogeneous materials, we first have to properly perform their segmentation. The segmentation consists in identifying and labelling all components of the microstructure. Mathematical morphology provides very efficient tools to perform this task.

Mathematical morphology is a theory for the analysis and processing of geometrical structures based on set theory and topology. Mathematical morphology finds most of its applications in the field of image processing and random structures simulation. The basic idea behind mathematical morphology is to analyze a set A of some topological space E by probing it with a compact set K, classicaly referred to as *structuring element*. Hence, mathematical morphology makes extensive use of classical operators of set theory, including for instance union or intersection.

The basic operators of mathematical morphology are dilation and erosion, which are defined in the following manner. Let A be a subset of $\mathbb{R}^n$. The dilated of the set A by the structuring element K is the set

$$A \oplus K = \{x \in \mathbb{R}^n | K_x \cap A \ \neq \emptyset\}, \tag{1.1}$$

where K_x is the translated of the compact K at $x \in E$. Similarly, the eroded of the set A by the structuring element K is the set

$$A \ominus K = \{x \in \mathbb{R}^n | K_x \subset A\}. \tag{1.2}$$

Dilation and erosion are dual operators with respect to the complement, in the sense that dilating the set A by the structuring element K is equivalent to erode A^c by K.

Dilation and erosion can be seen as the fundamental bricks of mathematical morphology, from which derive almost all other operators. For instance,

by combining erosion and dilation, we can define two new morphological operators. Let A, B be subsets of E. The *closing* A^B and the *opening* A_B of the set A by B are defined as follows:

$$A^B = (A \oplus \check{B}) \ominus B, \tag{1.3}$$

and

$$A_B = (A \ominus \check{B}) \oplus B. \tag{1.4}$$

More complicated operators can be defined to handle most low-level image processing tasks including denoising or pre-processing for segmentation.

Mathematical morphology also provide efficient techniques to perform image segmentation. Among these techniques, the most popular is the watershed algorithm, which has been applied in numerous case studies to perform material images segmentation. We introduce the key concepts of mathematical morphology in chapter 2. We refer the reader interested by a more extensive presentation of the theory to the books of Serra [33] and Soille [34].

1.2 Random set theory

Once segmented, the microstructure images still carry a huge amount of information. As a consequence, we need to develop mathematical techniques to capture the main features of the complex geometry under scrutiny. Mathematical set theory proves very useful in this regards. For instance, a natural way to describe inclusions in a matrix is to consider a set A, representing the included particles, and its complementary set A^c, representing the matrix. The study of such models falls into the scope of stochastic geometry. Stochastic geometry is a mathematical discipline which aims at providing a systematic description of random spatial patterns and whose development is intimately correlated to the one of mathematical morphology. The theory of random closed sets will be discussed in more details in chapter 2.

Let's go back to our previous example of inclusions in a matrix. As mentioned previously, to study the obtained sets, the idea behind mathematical morphology is to define a structuring element K (e.g a point, or a disk with some given diameter) and to use it to probe the subsets A and A^c. The simplest relations that one can build to study A and A^c are the following ones:

at a given point of the porous media, either the structuring element K hits the set A or it is disjoint from it. At each point, the knowledge of wheter or not some structuring element K hits a set A is sufficient to completely characterize A.

The deterministic approach exposed here proves however untrackable in practice. Obviously, processing each one of the microstructures constituting heterogeneous media is generally out of scope, since this would involve a considerable amount of data. However, considering the limited amount of information available on the material, it is interesting to rely on a statistical approach. From this perspective, instead of determining at each point if some structuring element intersects A, we will try to estimate the *probability* that the considered structuring element intersects A. A benefit of this approach is that we can determine statistical laws on experimental samples and generalize these laws to larger portions of the same material as long as the samples are statistically relevant.

In this framework, the set A representing the solid phase becomes completely characterized by the functional $T(K)$ defined for all structuring element K by

$$T(K) = P\{A \cap K \neq \emptyset\} = 1 - P\{K \cap A^c\}. \qquad (1.5)$$

$T(K)$ is called the Choquet capacity of the random closed set A. Note that the Choquet capacity is closely related to dilation and erosion operators. For all compact set $K \subset \mathbb{R}^n$, we have indeed

$$T(K) = P\{K \cap A \neq \emptyset\} = P\{x \in A \oplus \check{K}\} \qquad (1.6)$$

Through the Choquet capacity, mathematical morphology provides a solid mathematical framework to investigate the microstructure geometry. Since each compact set K brings its own information of the studied set A, the choice of structuring element allows one to conduct very specific statistical measurements on the random set A. For instance, if one chooses K to be a single point, the choquet capacity yieds

$$T(\{x\}) = P\{\{x\} \cap A \neq \emptyset\} = P\{x \in A\}, \qquad (1.7)$$

which is the *spatial law* of the set A. Similarly, if one chooses K to be the set $\{x, x+h\}$, the choquet capacity allows to calculate the *covariance* of the

random closed set.

$$T(\{x, x + h\}) = P\{x \in A, x + h \in A\}. \tag{1.8}$$

The covariance of the set A at a given point x and for a distance h is the probability that x and $x + h$ both belong to A. The covariance C_A provides useful information about the spatial arrangement of the random set A. In particular, it accounts for the presence of several scales in the studied set or for periodicity.

1.3 Random models for heterogeneous media

Mathematical morphology allows to analyse images of materials microstructures and to extract statistical features which characterize the microstructure in a very simple manner. Conversely, one is often interested in developing stochastic models of the microstructure that reproduce accurately some geometrical features of the original material.

The basic ingredients of stochastic geometry models are random point processes. A random point process $\mathcal{P}$ is a collection of random points of the space $\mathbb{R}^d$. A particular role is played by Poisson point processes. Let $\theta > 0$ be a positive real number. A Poisson point process on $\mathbb{R}^d$ is a point process such that the number $N(K)$ of points contained in any region K of $\mathbb{R}^d$ is a Poisson random variable with *intensity* θ:

$$P\{N(K) = k\} = \frac{\theta(K)^k}{k!} \exp(-\theta(K)). \tag{1.9}$$

Hence, adopting the point of view of mathematical morphology, we can generate a Poisson point process in a domain Ω of $\mathbb{R}^d$ by exploring Ω with a compact structuring element K and implanting a random number of points following the probability law (1.9) at each location. A general theory for random point processes is exposed in chapter 3.

Most classical models of stochastic geometry rely on Poisson point processes. An archetypal model is for instance the Boolean model, which is discussed in details in chapter 4. The Boolean model is a grain model which is obtained by implanting random primary grains A' on the germs $\{x_k\}$ of

a Poisson points process $\mathcal{P}$ with intensity θ. Note that primary grains can possibly overlap. The resulting set A is

$$A = \cup_{x_k \in \mathcal{P}} A'_{x_k}, \tag{1.10}$$

where A'_{x_k} denotes the translated of the primary grain A' at point x_k:

$$A'_{x_k} = \{x_k + y, \quad y \in A'\}. \tag{1.11}$$

Any shape can be used for the grain A', including convex, non-convex or even non connected sets.

Interestingly, the Boolean model is trackable analytically. Hence, one can easily prove that the number $N(K)$ of primary grains intersected by any compact region K follows a Poisson distribution of parameter $\theta\bar{\mu}(\check{A}' \oplus K)$:

$$P\{N = n\} = \frac{\theta^n \bar{\mu}(\check{A}' \oplus K)^n}{n!} \exp(-\theta\bar{\mu}(\check{A}' \oplus K))\}. \tag{1.12}$$

In this expression, $\bar{\mu}(\check{A}' \oplus K)$ denotes the average Lebesgue measure (the average surface in $\mathbb{R}^2$ or the average volume in $\mathbb{R}^3$) of a primary grain A' dilated by the compact set K. This result guarantees that the number of primary grains in any bounded window remains almost surely finite, and enables us to calculate the Choquet capacity of the boolean model

$$T(K) = 1 - \exp(-\theta\bar{\mu}(\check{A}' \oplus K)). \tag{1.13}$$

Knowing an analytical expression for the Choquet capacity, we can easily determine the covariance of the Boolean model that yields

$$C(h) = P\{x \in A, x + h \in A\} = 1 - \exp(-\theta\bar{\mu}(\check{A} \oplus l_h)), \tag{1.14}$$

where l_h denotes the structuring element constituted by both points 0 and h, h being some vector of $\mathbb{R}^n$.

Several models have been developed over the years that rely on similar approaches, including the dead leave model, Poisson flat processes or Boolean random functions. We elaborate on these models in chapters 5, 6, respectively. Other models have been specifically designed to simulate microstructures of crystals, which notably include Voronoï and Johnson-Mehl tessellations. The rather difficult study of these models is the object of chapter 6.
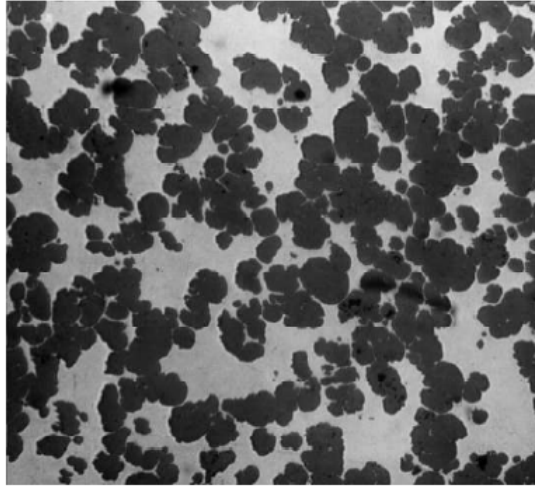
Figure 1.1: Fe-Ag composite microstructure. The microstructure can be modeled by a Boolean model of spheres (see Chapter 4). This illustration is taken from the lecture notes of Jeulin [12]
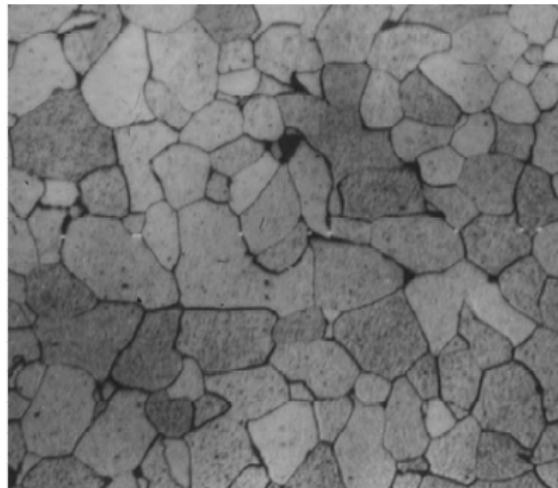
Figure 1.2: Polycristal microstructure. The microstructure can be modeled by a random tessellation of $\mathbb{R}^2$ (see Chapter 5). This illustration is taken from the lecture notes of Jeulin [12]

Part I

Theory

Chapter 2

Fundamentals of random set theory

2.1 Introduction

The study of heterogeneous media can easily be formalized through mathematical set theory. For instance, a natural way to describe a porous media is to consider the union of a set A representing the solid component and of its complement A^c representing the porous network. Similarly, inclusions in a matrix can be described by a set A, representing the included particles, and its complementary set A^c, representing the matrix.

To study the obtained sets, the idea behind mathematical morphology is to define a structuring element K and to use it to probe the subsets A and A^c. The simplest relations that one can build to study A and A^c are the following ones: at a given point of the porous media, either the structuring element K hits the set A or it is disjoint from it *i.e* included in the complementary set. At each point, the knowledge of wheter or not some structuring element K hits a set A is sufficient to completely characterize A. Moreover, relying on intersection, one can define a complete topology on the set $\mathcal{P}(E)$ of all subsets of E. We elaborate on this topic in appendix A1.

The deterministic approach exposed here proves however difficult. Obviously, studying each one of the microstructures constituting heterogeneous media is generally out of scope, since this would require processing a consid-

erable amount of data. Therefore, it is often interesting to adopt a statistical approach. From this perspective, at a given point, the information is not anymore wheter or not some structuring element hits the set A but the *probability* that this structuring element hits A.

A benefit of this approach is that statistical laws can be determined on experimental samples providing us with partial knowledge of the microstructure. This opens the way to the development of predictive models aimed at simulating random structures. Such models have been successfully used to simulate microstructures of a wide range of media and textures. Dispersions of small particles in a matrix can for instance be modelled by realizations of stochastic point processes (see Chapter 3). Similarly, multiphase media can be simulated by multicomponent random sets.

2.2 Facts from mathematical morphology

Mathematical morphology is a theory for the analysis and processing of geometrical structures. It is most commonly applied to digital images, but it can be employed as well on graphs, surface meshes, solids, and many other spatial structures. Random sets theory makes an extensive use of the concepts of mathematical morphology. It is therefore natural to start this introduction with some concepts of mathematical morphology.

2.2.1 Dilation and erosion

The basic idea behind mathematical morphology is to analyse a set A of some topological space E by probing it with a compact set K (structuring element). Hence, mathematical morphology makes extensive use of classical operators of set theory, including for instance union or intersection. We first introduce the two basics bricks of mathematical morphology, namely erosion and dilation.

Definition 2.2.1 *Let A be a closed set in E. The dilated of the set A by the structuring element K is the set*

$$D^K(A) = \{x \in E | K_x \cap A \; \neq \emptyset\}, \tag{2.1}$$

where K_x is the translated of the compact K at $x \in E$. Similarly, the eroded of the set A by the structuring element K is the set

$$E^K(A) = \{x \in E | K_x \subset A\}. \tag{2.2}$$

Dilation and erosion are dual operators with respect to the complement, in the sense that dilating the set A by the structuring element K is equivalent to erode A^c by K.

We assume now that E is the euclidean space $\mathbb{R}^d$ of dimension d. The vectorial space structure of $\mathbb{R}^d$ allows us to define new operations on $\mathcal{P}(\mathbb{R}^d)$, namely the Minkowski addition and substraction.

Definition 2.2.2 *Let A and B be subsets of $\mathbb{R}^d$. The Minkowski addition is defined by*

$$A \oplus B = \{a + b, a \in A, b \in B\} \tag{2.3}$$

The Minkowski addition is an associative and commutative operation. Note that $(\mathcal{P}(\mathbb{R}^d), \oplus)$ is an abelian semi-group, whose neutral element is $\{0\}$.

We introduce some notations at this point. Let x be a point of $\mathbb{R}^d$. We denote by A_x the set A translated at point x:

$$A_x = A \oplus \{x\}. \tag{2.4}$$

Similarly, we denote by $\check{B}$ the symetric set of $B \in \mathcal{P}(E)$ defined by

$$\check{B} = \{-x, x \in B\} \tag{2.5}$$

Definition 2.2.3 *Using these notations, we can define the Minkowski substraction by duality. Let A and B be subsets of E. The Minkowski substraction is defined by*

$$A \ominus B = (A^c \oplus B)^c = \cap_{x \in B} A_x \tag{2.6}$$

We can also express the classical dilation and erosion operators of mathematical morphology as functions of the Minkowski addition and substraction respectively.

Definition 2.2.4 *Let A and B be subsets of $\mathbb{R}^d$. The erosion of A by B yields the set*

$$\{x \in E, B_x \in A\} = A \ominus \check{B}. \tag{2.7}$$

Similarly, we can check by duality that the dilation of A by B yields the set

$$\{x \in E, B_x \cap A \neq \text{ø}\} = A \oplus \check{B}. \tag{2.8}$$

Problem 2.2.1 *Let $A, B \in \mathcal{P}(\mathbb{R}^d)$ be subsets of $\mathbb{R}^d$, and K, K_1, K_2 be compact sets of $\mathcal{K}(\mathbb{R}^d)$. Show that*

$$(A \ominus \check{K}_1) \ominus \check{K}_2 = A \ominus (\check{K}_1 \oplus \check{K}_2), \tag{2.9}$$

$$(A \cap B) \ominus \check{K} = (A \ominus \check{K}) \cap (B \ominus \check{K}), \tag{2.10}$$

and

$$A \oplus (\check{K}_1 \cup \check{K}_2) = (A \oplus \check{K}_1) \cup (A \oplus \check{K}_2). \tag{2.11}$$

Problem 2.2.2 *Let $A \in \mathcal{P}(\mathbb{R}^d)$ be a subset of $\mathbb{R}^d$, and K be a compact set of $\mathcal{K}(\mathbb{R}^d)$.*
1. Show that if A is convex, then $A \ominus \check{K}$ is convex.
2. If A is convex, under which assumption is $A \oplus \check{K}$ convex?

2.2.2   Opening and closing

By combining erosion and dilation, we can define two new morphological operators. Let $A, B \in \mathcal{P}(E)$ be subsets of E. The *closing* A^B and the *opening* A_B of the set A by B are defined as follows:

$$A^B = (A \oplus \check{B}) \ominus B, \tag{2.12}$$

and

$$A_B = (A \ominus \check{B}) \oplus B. \tag{2.13}$$

The opening and closing operators are widely used in mathematical morphology. These operator can for instance be used to perform image denoising and are the fundamental bricks upon which builds most of the theory.

2.2.3   Granulometry

A first application of openings and closings related to the description of random sets are the granulometry operators. Intuitively, a granulometry by closing (resp. by opening) is a family of closing (resp. opening) of increasing sizes which allows us to characterize the size distribution of the connected components of any random set.
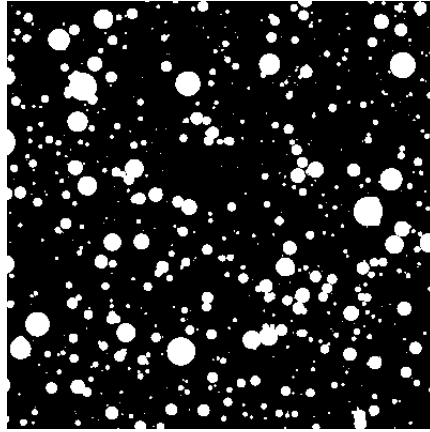
Figure 2.1: Segmented image of some microstructure simulated with the software VtkSim [7] with a Boolean model of spheres.
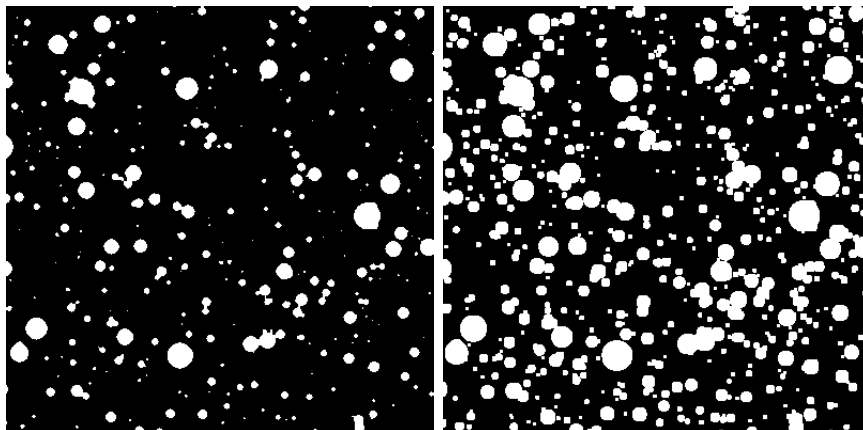
Figure 2.2: Erosion (left) and dilation (right) of the segmented image with a disk of radius one pixel. Erosion removes the smallest components of the microstructure.
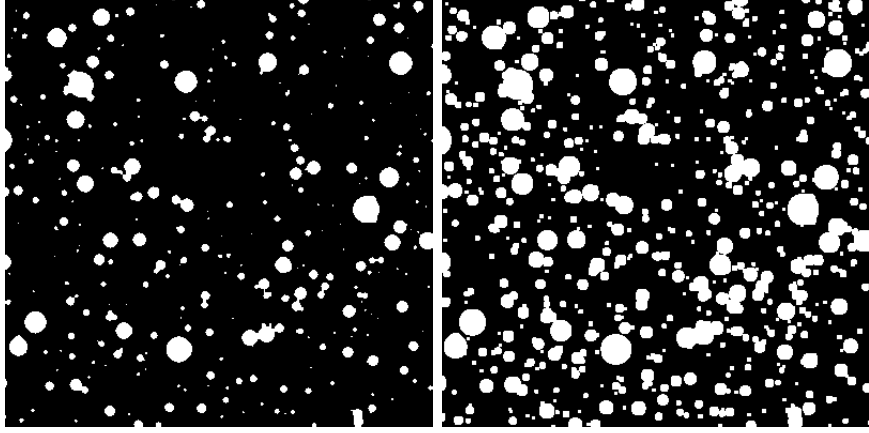
Figure 2.3: Opening (left) and closing (right) of the segmented image with a disk of radius one pixel.

Definition 2.2.5 *More formally, a granulometry is a family of set operators Φ_λ depending on a positive parameter λ satisfying the following properties:*
i) For all A in $\mathcal{F}(E)$, $\Phi_\lambda(A) \subset A$: Φ_λ is anti-extensive.
ii) If $A \subset B$, then $\Phi_\lambda(A) \subset \Phi_\lambda(B)$: Φ_λ is increasing.
ii) $\Phi_\lambda \circ \Phi_\mu = \Phi_\mu \circ \Phi_\lambda = \Phi_{\max(\mu,\lambda)}$

The axiomatic of granulometries was first formulated by Matheron in [20]. Note that an immediate consequence of the last point of the definition is that Φ_λ is necessarily an idempotent operator, in the sense that $\Phi_\lambda \circ \Phi_\lambda = \Phi_\lambda$.

As stated above, the axiomatic of granulometries remains very general. In practice, we will often consider granulometries relying on a familly of openings. Let K be a convex set. We consider the family $\{K_\lambda, \lambda > 0\}$, where $K_\lambda = \lambda K$. The operator

$$\Phi_\lambda(A) = (A \ominus \check{K}_\lambda) \oplus K_\lambda, \tag{2.14}$$

defined for all closed set A of $\mathcal{F}(E)$, is a granulometry. For a random set A, a granulometry by openings describes the size distribution of the elements of A by opening by convex sets.

In a similar manner, we can define a granulometry by closing. To that end, we consider the operator

$$\Phi_\lambda(A) = (A \oplus \check{K}_\lambda) \ominus K_\lambda, \tag{2.15}$$

defined for all closed set A of $\mathcal{F}(E)$. A granulometry by closing describes the size distribution of the elements of A by closing by convex sets.

2.3 Probabilistic approach and Choquet capacity

Concepts of mathematical morphology prove very convenient to study random sets. In particular, it is of interest to translate some compact set K in an observation window to analyse a random closed set A of $\mathbb{R}^n$. Two elementary events can occur: if $K \cap A = \emptyset$, the structuring element K is disjoint from A; otherwise, if $K \cap A \neq \emptyset$, the structuring element K hits the set A. The random closed set A is completely characterized by the functional $T(K)$ defined for all compact sets K by

$$T(K) = P\{A \cap K \neq \emptyset\} = 1 - P\{K \cap A^c\} = 1 - Q(K) \tag{2.16}$$

$T(K)$ is called the Choquet capacity of the random closed set A. Note that the Choquet capacity is closely related to dilation and erosion operators. For all compact set $K \subset \mathbb{R}^n$, we have indeed

$$T(K) = P\{K \cap A \neq \emptyset\} = P\{x \in A \oplus \check{K}\} \tag{2.17}$$

Problem 2.3.1 *Show that*

$$Q(K) = P\{K \subset A^c\} = P\{x \in A^c \ominus \check{K}\}. \tag{2.18}$$

The structuring element K can be a single point $\{x\}$ of $\mathbb{R}^n$ or any compact set of $\mathbb{R}^n$. However, we have to insist on the fact that the choice of structuring element is fundamental. Each compact set K indeed brings its own information on the studied set A. For instance, if one chooses K to be a single point, the choquet capacity yieds

$$T(\{x\}) = P\{\{x\} \cap A \neq \emptyset\} = P\{x \in A\}, \tag{2.19}$$

which is the *spatial law* of the set A. Similarly, if one chooses K to be the set $\{x, x + h\}$, the choquet capacity allows to calculate the *covariance* of the random closed set.

$$T(\{x, x + h\}) = P\{x \in A, x + h \in A\}. \tag{2.20}$$

2.3.1   Covariance

The covariance is a - if not *the* - fundamental tool to describe spatial arrangement in a random closed set.

Definition 2.3.1 *The covariance of a random set $A \subset \mathbb{R}^n$ is the function C_A defined on $\mathbb{R}^n \times \mathbb{R}^n$ by*

$$C_A(x, x + h) = P\{x \in A, x + h \in A\}, \tag{2.21}$$

where h is some vector of $\mathbb{R}^n$.

The covariance of the set A at a given point x and for a distance h is the probability that x and $x + h$ both belong to A. Note that for a stationary random set, the covariance is a function of the distance h only:

$$C_A(x, x + h) = C_A(h). \tag{2.22}$$

If in addition the set A is ergodic, the covariance $C(h)$ can be estimated from the volume fraction of $A \cap A_{-h}$ to be

$$C_A(h) = P\{x \in A \cap A_{-h}\} = V(A \cap A_{-h}) = V(A \ominus \check{h}), \tag{2.23}$$

where h is the set $\{x, x + h\}$. In practice, the covariance is usually estimated from experimental samples of the studied random set using equation (2.23).

The covariance C_A provides useful information about the spatial arrangement of the random set A. In particular, it accounts for the presence of several scales in the studied set or for periodicity. Note that by definition, $C_A(0)$ simply corresponds to the volumic fraction of the set A. For any orientation, the covariance $C(h)$ reaches a sill at the distance or range h_∞. At this distance, events $\{x \in A\}$ and $\{x + h_\infty\}$ are independent and we have

$$C_A(h_\infty) = p^2, \tag{2.24}$$

. These considerations enable us to define a normalized version of the covariance that remains between 0 and 1:

$$\gamma(h) = \frac{C(h) - p^2}{p(1 - p)}. \tag{2.25}$$

For an ergodic experimental sample, the covariance can be estimated with relation (2.23). An alternative approach is to use the Fourier transform, as demonstrated in the following problem.

Problem 2.3.2 *Let A be a subset of $\mathbb{R}^d$. Show that the covariance of A is*

$$C_A(h) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^\varkappa} |\hat{f}(\xi)|^2 \exp(i\xi h) d\xi. \tag{2.26}$$

The covariance of a random set A generally depends on the orientation of the vector h. The *isotropised covariance* is defined to be

$$\bar{C}(h) = \int_{S_{d-1}} C(h\mathbf{u}) U(d\mathbf{u}), \tag{2.27}$$

where u is an unitary vector and $U(d\mathbf{u})$ denotes the uniform distribution on the unit sphere S_{d-1}.

Some features of the covariance can easily be expressed analytically. A fundamental example is given by its first derivative.

Proposition 2.3.3 *Let A be a random set on $\mathbb{R}^d$. Then, the first derivative of the covariance is*

$$\frac{d}{dh} C_A(h\mathbf{u}) = -\lim_{h \to 0} \nu_{d-1}((K \cap K_{r\mathbf{u}})|_{\mathbf{u}^\perp}), \tag{2.28}$$

where ν_{d-1} is the Lebesgue measure on $\mathbb{R}^{d-1}$ and $\mathbf{u}$ is some unit vector. $(K \cap K_{r\mathbf{u}})|_{u\perp}$ denotes the projection of $(K \cap K_{r\mathbf{u}})$ on the hyperplane that has $\mathbf{u}$ as normal vector.

When $d = 3$, for the isotropic case, equation (2.28) simply yields

$$\frac{dC_A}{dh}(0) = -S(A), \tag{2.29}$$

where $S(A)$ is the surface area of the set A in $\mathbb{R}^3$. Similarly, when $d = 2$, equation (2.28) yields

$$\frac{dC_A}{dh}(0) = -\mathcal{P}(A), \tag{2.30}$$

where $\mathcal{P}(A)$ is the length of the perimeter of A in $\mathbb{R}^2$.
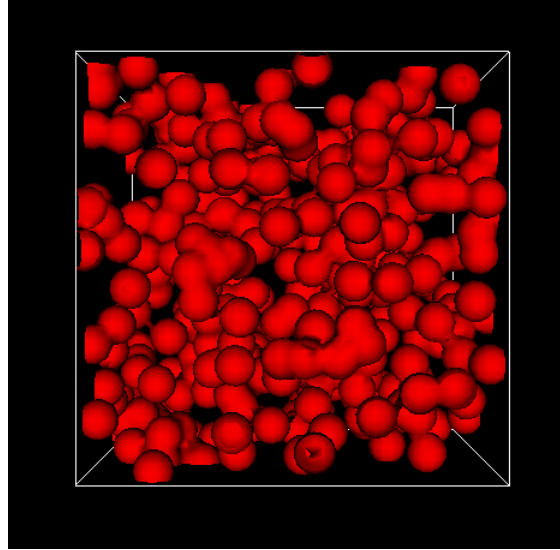
Figure 2.4: Random realization $\mathcal{B}$ of a Boolean model of spheres of constant radius (see chapter 3), obtained with the software VtkSim [?].
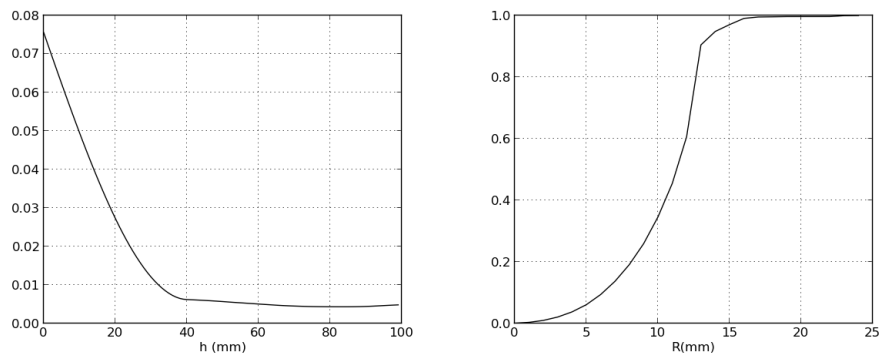
Figure 2.5: Set covariance (left) and cumulative granulometry by opening (right) of the set $\mathcal{B}$.

2.4   Measurements on random sets

Starting from a material image, it is possible to measure a countless number of parameters. It is however essential for these parameters to be significant regarding the physics and the geometry of the material. Two families of parameters are usually considered, namely *metric* and *topological* parameters. Intuitively, if we consider inclusions in a matrix, we can be interested in knowing the volume fraction of inclusions. This parameter is metric, in the sense that it is directly obtained by a measurement. Conversely, we could also be interested in knowing the number of included particles. This parameter is topological, in the sense that it is obtained by counting.

In practice, strict limitations are imposed on the measurements. First, one obviously wants the measurement to be isometry-invariant. It is indeed important to ensure that a measurement conducted on a set X is independent from the location of the set X. An homogeneity condition must also be satisfied. If a measurement is conducted at different scales on the same set X, one obviously want to get the same results. This conditions yields, for some measurement W,

$$W(\lambda X) = \lambda^d W(X), \tag{2.31}$$

where $d \in \mathbb{N}$ and $\lambda > 0$. An additional requirement is for the measurement to be additive. In mathematical terms, this condition is expressed through the relation

$$W(X) + W(Y) = W(X \cup U) + W(W \cap Y). \tag{2.32}$$

A final requirement is for the measurement to be continuous. A small deformation of the measured set cannot result in large fluctuations of the measurements.

Finite union of convex sets play a key role in stochastic geometry. Most of the geometrical theory of random builds upon results obtained for convex sets and on their generalization on finite unions of convex sets. A fundamental result states that all measurements satisfying the conditions enumerated above can be expressed as a linear combination of a finite number of functionals, referred to as Minkowski's functionals.

2.4.1   Minkowski functionals and intrinsic volumes

Definition 2.4.1 *A subset C of $\mathbb{R}^d$ is said to be convex if for every pair of points x, y in C and every $c \in [0,1]$, we have $cx + (1-c)y \in C$.*

Affine linear subspaces are archetypal examples of convex sets. An affine linear subspace L is indeed characterized by the property that for all points $x, y \in L$, $cx + (1-c)y \in L$ for all $c \in \mathbb{R}$. An affine linear subspace of dimension k of $\mathbb{R}^d$ is defined by the implicit equation

$$\{x \in \mathbb{R}^d, c_0 + c_1 x_1 + c_2 x_2 + ... + c_k x_k = 0, \sum_i^k c_i = 1\}, \tag{2.33}$$

where $c_i \in \mathbb{R}$ for all i. k-dimensional affine linear subspaces are referred to as $k-$flats or $k-$planes.

Definition 2.4.2 *We call convex bodie any compact convex subset of $\mathbb{R}^d$, and we denote by $\mathcal{C}(\mathbb{R}^d)$ the system of all convex bodies of the d-dimensional euclidean space $\mathbb{R}^d$.*

Definition 2.4.3 *A convex body functional is a functional h defined on $\mathcal{C}(\mathbb{R}^d)$ which assigns a real value $h(C)$ to each $C \in \mathcal{C}(\mathbb{R}^d)$. A convex body functional is said to be:*
- isometry-invariant if for all isometry $\mathcal{G}$, we have $h(\mathcal{G}C) = h(C)$.
- monotone if $C_1 \subset C_2$ implies $h(C_1) \leq h(C_2)$.
- C-additive if for all pairs of convex bodies C_1, C_2 satisfying $C_1 \cup C_2 \in \mathcal{C}(\mathbb{R}^d)$, we have $h(C_1) + h(C_2) = h(C_1 \cup C_2) + h(C_1 \cap C_2)$.

A fundamental theorem of integral geometry states that all convex body functional that are isometry-invariant, monotone and $C-$additive can be expressed as linear combinations of the Minkowski functionals W_d. The Minkowski functionals are isometry-invariant, monotone, C-additive convex body functionals, defined directly on $\mathcal{C}(\mathbb{R}^d)$ by the formula

$$W_k(C) = \frac{b_d}{b_{d-k}} \int_{\mathbb{L}_k} \mu_{d-k}(C|_{E^\perp}) U_k(dE). \tag{2.34}$$

In this expression, b_k denotes is the volume of the unit ball in $\mathbb{R}^k$. μ_k is the k-dimensional Lebesgue measure. $\mathbb{L}_k$ is the set of all k-subspaces, $C|_{E^\perp}$ is the

orthogonal projection of the convex body C on $E^\perp$, $E^\perp$ is the $(d-k)$-subspace orthogonal to $E \in \mathbb{L}_k$, and U_k is the uniform probability distribution on $\mathbb{L}_k$.

$\forall d > 0$, for $k = 0$, equation (2.34) becomes

$$W_0(C) = \int_{\mathbb{L}_0} \mu_d(C|_{E^\perp})U_0(dE) = \mu_d(C). \qquad (2.35)$$

Hence, $W_0(C)$ is equal to the volume $\mu_d(C)$.
Similarly, $\forall d > 0$, for $k = d$, we find

$$W_d(C) = b_d \qquad (2.36)$$

Theorem 2.4.1 *Every non-negative, motion invariant, monotone, $C-$additive convex body functional h on $\mathcal{C}(\mathbb{R}^d)$ can be expressed as a linear combination of the Minkowski functionals.*

$$h(C) = \sum_{k=0}^{d} W_k(C), \quad C \in \mathcal{C}(\mathbb{R}^d). \qquad (2.37)$$

This result is know as Hadwiger's characterization theorem.

In the particular cases of $d = 1, 2, 3$, Minkowski functionals yield

d = 1	$W_0(C) = \mathcal{L}(C), \quad W_1(C) = 2$
d = 2	$W_0(C) = \mathcal{A}(C), \quad W_1(C) = \dfrac{L(C)}{2},$ $W_2(C) = \pi$
d = 3	$W_0(C) = \mathcal{V}(C), \quad W_1(C) = \dfrac{S(C)}{3},$ $W_2(C) = \dfrac{M(C)}{3}, \quad W_3(C) = \dfrac{4\pi}{3},$

where

- $\mathcal{L}(C)$ is the *length* of the convex body C when $d = 1$,

- $\mathcal{A}(C)$ is the *area* of the convex body C when $d = 2$,

- $\mathcal{V}(C)$ is the *volume* of the convex body C when $d = 3$,

- $L(C)$ is the *boundary length* of the convex body C when $d = 2$,

- $S(C)$ is the *surface area* of the convex body C when $d = 3$.

When $d = 3$, the quantity $M(C)$ is defined as the integral of mean curvature. It is defined by

$$M(C) = \int_{\partial C} m(x)dS, \tag{2.38}$$

where

$$m(x) = \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \tag{2.39}$$

is the mean curvature at location x of the surface.

Some authors prefer relying on an equivalent family of functionals, namely the intrisic volumes, rather than on Minkowski functionals. Intrinsic volumes are non-negative, motion invariant, monotone, $C-$additive convex body functional on $\mathcal{C}(\mathbb{R}^d)$ that are related to the Minkowski functionals W_k through the relation

$$b_{d-k}V_k(C) = \binom{d}{k}W_{d-k}(K), \tag{2.40}$$

for $k = 0, 1, .., d$. In the particular cases of $d = 1, 2, 3$, intrinsic volumes yield

d = 1	$V_0(C) = 1, \quad V_1(C) = \mathcal{L}(C)$
d = 2	$V_0(C) = 1, \quad V_1(C) = \dfrac{L(C)}{2},$ $V_2(C) = \mathcal{A}(C)$
d = 3	$V_0(C) = 1, \quad V_1(C) = \dfrac{M(C)}{\pi},$ $V_2(C) = \dfrac{S(C)}{2}, \quad W_3(C) = \mathcal{V}(C).$

2.4.2 Steiner formulae

Convex geometry is intimately correlated to mathematical morphology. In particular, Steiner formulae provide a useful tool to express the volume of a convex body C dilated by a ball of finite radius $r > 0$ in terms of the Minkowski functionals of C.

Definition 2.4.4 *Let A be a set in $\mathbb{R}^d$. The parallel set of distance r of A is the set $A_{\oplus r} = A \oplus B(0, r)$.*

Let C be a convex body in $\mathbb{R}^d$. The following fundamental result allows us to calculate the volume of the parallel convex body of distance r of C knowing the Minkowski functionals of C.

Theorem 2.4.2 *Let C be in $\mathcal{C}(\mathbb{R}^d)$ and $r > 0$. We have:*

$$\mu_d(C \oplus B(0, r)) = \sum_{k=0}^{d} \binom{d}{k} W_k(C) r^k. \tag{2.41}$$

This formula is classicaly referred to as Steiner formula. Similarly, with the intrinsic volumes, we obtain:

$$\mu_d(C \oplus B(0, r)) = \sum_{k=0}^{d} b_{d-k} V_k(C) r^{d-k}, \tag{2.42}$$

where b_{d-k} is the volume of the unit ball of $\mathbb{R}^{d-k}$.

2.4.3   Stereology and Crofton formulae

We must keep in mind the fact that an image is often the bidimensional representation of a tridimensional microstructure. Therefore, we will often have to estimate metric or topological parameters in $\mathbb{R}^3$ from parameters measured in $\mathbb{R}^2$. A parameter is said to be *stereological* if it can be used in this purpose.

An important result in stereology is given by the Crofton formula, which allows us to calculate the intrinsic volumes for intersections of convex bodies with flats. We refer the reader interested by more details on stereology to the books of Stoyan, Kendall and Mecke [4] and Schneider and Weil [32].

2.5   Notes

This chapter provides a short introduction to basic concepts in stochastic geometry, integral geometry and mathematical morphology. More extensive introductions can be found in the treaty of Matheron published in 1975 [20] or in the more recent book of Stoyan, Kendall and Mecke [4]. Integral geometry is covered in depth in the book of Schneider and Weil [32]. The lecture notes of Jeulin [12] are another useful reference for the material covered in this chapter.

Mathematical morphology is treated in depth in the books of Serra [33], Soille [?] or Coster and Chermant [6] (in French).

Chapter 3

Point processes

In this chapter, we present the general theory of random point processes. A random point process $\mathcal{P}$ is a collection of random points. Point processes can be considered as the basic ingredients of stochastic geometry. A particular role is played by Poisson point processes in the d-dimensional space $\mathbb{R}^d$.

In section 3.1, we introduce a general framework for the study of point processes. We study more specifically Poisson point processes in section 3.2. Marked point processes are then discussed in section 3.3.

3.1 General theory

In this section, we introduce a general framework for the study of point processes on locally compact topological spaces. The results of this section are technical and we will only state the most relevant ones, often without proof. We refer the reader to the reference textbooks of Weil and Schneider [32] and Stoyan *et al.* [4] for a more extensive presentation.

3.1.1 Random point processes as counting measures

Let E be a locally compact space with a countable topological basis. We denote by $\mathcal{B}(E)$ the Borel σ-algebra of E. Let $M(E)$ be the set of all locally finite measures defined on E. Recall that a measure η is said to be locally finite if for all compact C in $\mathcal{K}(E)$, $\eta(C) < \infty$. For all borelian set A in $\mathcal{B}(E)$, we define the evaluation map

$$\Phi_A : M \to \mathbb{R} \cup \{\infty\}. \tag{3.1}$$

When equipped with the σ-algebra $\mathcal{M}$ generated by all evaluation maps $\{\Phi_A, A \in \mathcal{B}(E)\}$, $M(E)$ forms a measurable space.

A class of measures of particular interest for the study of point processes is provided by the counting measures.

Definition 3.1.1 *A counting measure on E is a measure η in $M(E)$ such that for all borelian set A in $\mathcal{B}(A)$, $\eta(A) \in \mathbb{N} \cup \{\infty\}$. We denote by $N(E)$ the set of all counting measures on E.*

It can be shown that $N(E)$ is a measurable subset of $(M(E), \mathcal{M})$ [32]. We denote by $\mathcal{N}$ the corresponding σ-algebra. A fundamental example of counting measure is given by locally finite sums of Dirac measures:

$$\eta = \sum_{k=1}^{n} \delta_{x_k}. \tag{3.2}$$

Another example is given by random Poisson counting measures. For all borelien set A of $\mathbb{R}^d$, a random Poisson counting measure follows a Poisson distribution given by

$$\eta(A) = \frac{\Theta(A)^k}{k!} \exp(-\Theta(A)), \tag{3.3}$$

where Θ is some(real) measure on the σ-algebra $\mathcal{B}(\mathbb{R}^d)$.

Point processes can be apprehended either as random sets of discrete points or as random counting measures giving the number of points contained in any domain of E. For a counting measure $\eta \in N(E)$, the support supp η is the smallest closed set A in E such that $\eta(E/A) = 0$. The mapping $\eta \to$ supp η identifies a random measure to its corresponding point process. As alluded to earlier, the set of all locally finite measures on E can be equipped with a σ-algebra. This consideration enables us to define a probability law on $M(E)$.

Definition 3.1.2 *A random measure X on E is a measurable map from some probability space $\{\Omega, A, \mathbb{P}\}$ into the measurable space $\{M(E), \mathcal{M}\}$. The image measure $\mathbb{P}_X$ is the distribution of X.*

For a random measure X which is almost surely concentrated on $N(E)$, since $N(E)$ is a measurable subset of $M(E)$, supp X is a random point process on E. Its distribution is defined for all $Y \in \mathcal{N}$ by the probabilities

$$P(Y) = \mathbb{P}\{X \in Y\} = \mathcal{P}\{\omega \in \Omega, X(\omega) \in Y\}. \tag{3.4}$$

The finite-dimensional distributions are of particular interest. They are defined for any family $\{B_1, B_2, .., B_k\}$ of bounded Borel sets of E to be the probabilities

$$\mathbb{P}\{X(B_1) = n_1, .., X(B_k) = n_k\}, \tag{3.5}$$

where $n_1, .., n_k$ are positive integers.

3.1.2 Intensity measure

From now on, we will assume E to be the d-dimensional Euclidean space $\mathbb{R}^d$.

Definition 3.1.3 *The intensity of the random measure X is the measure on $\mathbb{R}^d$ defined for all borelian set A in $\mathcal{B}(\mathbb{R}^d)$ by*

$$\Theta(A) = \mathbb{E}[X(A)]. \tag{3.6}$$

The intensity measure of a random point process can be seen as the equivalent of the mean of a real-valued random variable. It is of interest to consider the particular case of a stationary point process. A point process is said to be stationary if its distribution is invariant by translation. Hence, for any configuration Y in $\mathcal{N}$ and for $x \in \mathbb{R}^d$, we have

$$\mathbb{P}\{X \in Y\} = \mathbb{P}\{X + x \in Y\}.$$

For a stationary point process, the intensity measure is necessarily translation-invariant. It implies that

$$\theta(B) = \lambda\mu_D(B), \tag{3.7}$$

where μ_D is the d-dimensional Lebesgue measure on $\mathbb{R}^d$ and λ some positive real number.

We conclude this section by stating the Campbell theorem, whose proof is left as exercise.

Theorem 3.1.1 *Let X be a random measure on E with intensity measure Θ, and let $f : E \to \mathbb{R}$ be a non-negative, measurable function. Then, we have*

$$\mathbb{E}\left[\int_E f dX\right] = \int_E f d\Theta. \tag{3.8}$$

3.2   Poisson point process

3.2.1   Definition and characterization

Definition 3.2.1 *Let θ be a locally finite measure on $\mathbb{R}^d$. A Poisson point process on $\mathbb{R}^d$ is a point process such that the number $N(K)$ of points contained in any compact K of $\mathbb{R}^d$ is a Poisson random variable with parameter $\theta(K)$:*

$$P\{N(K) = k\} = p_k(K) = \frac{\theta(K)^k}{k!} \exp(-\theta(K)), \qquad (3.9)$$

where the intensity θ is defined by

$$\theta(K) = \int_K \theta(dx). \qquad (3.10)$$

Problem 3.2.1 *Show that the probability generating function $G_K(s)$ of the random variable $N(K)$ is*

$$G_K(s) = \sum_{k=0}^{+\infty} p_k(K)s^k = \exp[\theta(K)(s-1)]. \qquad (3.11)$$

An important consequence of definition 3.2.1 is that for any family $\{K_i, i \in I\}$ of disjoint compact sets, the random variables $N(K_i)$ are independant. This property is referred to as *complete independance*. In many practical situations, the measure θ is proportional to the Lebesgue measure on the σ-algebra of $\mathbb{R}^d$. In this case, the Poisson point process is said to be *stationary* and the number $N(K)$ of points contained in a given compact K is

$$P\{N(K) = k\} = \frac{(\theta \mu_d(K))^k}{k!} \exp(-\theta \mu_d(K)), \qquad (3.12)$$

μ_d being the Lebesgue measure of $\mathbb{R}^d$.

A Poisson point process is easily characterized by its Choquet capacity, as demonstrated below in proposition 3.2.2.

Proposition 3.2.2 *The Choquet capacity $T(K)$ of a Poisson point process is*

$$T(K) = 1 - P\{N(K) = 0\} = 1 - \exp(-\theta(K)). \qquad (3.13)$$

If the process is stationary, the Choquet capacity becomes

$$T(K) = 1 - \exp(-\theta\mu_d(K)). \tag{3.14}$$

Proof By definition, the Choquet capacity $T(K)$ of a Poisson point process is the probability that K intersects at least one point of the process. According to definition 3.2.1, we have

$$P\{N(K) > 0\} = 1 - P\{N(K) = 0\} = 1 - \exp(-\theta(K)). \tag{3.15}$$

For a stationary Poisson point process X, the intensity can easily be estimated from some experimental dataset by

$$\bar{\theta} = \frac{X(W)}{\mu_d(W)}, \tag{3.16}$$

where W denotes the observation window in $\mathbb{R}^d$. As the size of the window increases, we have $\bar{\theta} \to \theta$.

Problem 3.2.3 *Show that if $\mathcal{P}_1$, ... $\mathcal{P}_n$ are n independant Poisson point processes $\mathcal{P}_k$ with respective intensities θ_1, .., θ_n, then the union set $\mathcal{P} = \cup_{k=0}^{n}\mathcal{P}_k$ is a Poisson point process of intensity $\theta = \sum_{k=1}^{n} \theta_k$.*

3.2.2   Simulation of a stationary Poisson point process

We elaborate in this paragraph on the practical implementation of a stationary Poisson point process in a domain $\mathcal{D}$ of $\mathbb{R}^d$. The simulation can be performed in two stages. First, we simulate a Poisson random variable that defines the number of points implanted on the domain $\mathcal{D}$. Then, we simulate the required number of points in the domain according to an uniform law.

Random Poisson variables can be generated from the uniform law. Hence, if U is a random variable drawn from an uniform law on $[0, 1]$, then $-\ln(U)$ is as required.

For the second step of the simulation, it is straightforward to simulate a random point uniformly distributed in $[0, 1]^d$. For any hypercube domains, the simulation can be first performed on $[0, 1]^d$ and then be translated and

scaled to produce a sequence of points in the hypercube. For more complicated domains, simulation of uniform random points can be tackled through rejection sampling or approximation. Rejection sampling consists in defining a rectangle $\mathcal{R}$ that contains the domain $\mathcal{D}$. A sequence of independent uniformly random points is simulated in $\mathcal{R}$ until a first point falls into $\mathcal{D}$. The process is repeated until the required number of points has been reached in the domain $\mathcal{D}$. Approximation consists in replacing the domain $\mathcal{D}$ by a union of open squares that approximate $\mathcal{D}$.

3.2.3 Cox-Poisson point processes

Definition 3.2.2 *Let θ be a locally random finite measure on $\mathbb{R}^n$. A Cox-Poisson point process on $\mathbb{R}^n$ is a point process such that the number $N(K)$ of points contained in any compact K of $\mathbb{R}^n$ is a Poisson random variable with parameter $\theta(K)$:*

$$P\{N(K) = k\} = p_k(K) = \frac{\theta(K)^k}{k!} \exp(-\theta(K)), \qquad (3.17)$$

where the intensity θ is the random variable defined by

$$\theta(K) = \int_K \theta(dx). \qquad (3.18)$$

Cox-Poisson point processes are an extension of Poisson point processes in the sense that for these processes, the intensity θ is a random variable.

A fundamental example of Cox point processes are Poisson point processes restricted to some random closed set. Let A be a random closed set of $\mathbb{R}^n$, an $\lambda > 0$. The measure

$$\theta(K) = \int_K \theta 1_A(x)dx, \qquad (3.19)$$

where 1_A is the indicative function of the set A defines a Cox-Poisson point process. This Cox-Poisson point process can be seen as the restriction of a stationary Poisson point process of intensity θ to the random closed set A. Such point processes are often used in stochastic geometry to construct multiscale models. We refer the reader interested by these models to the paper of Jeulin [12].

3.2.4 Hard-core point processes

A hard-core point process is a point process for which the points cannot lie closer than a specified distance D. Let $\mathcal{P}$ be an homogeneous Poisson point process with intensity θ. We can obtain a hard-core point process by thinning. Thinning consists in deleting points from the point process according to some rules. In practice, for some domain Ω, we first generate the Poisson variable N that indicates the number of points implanted in the domain. Then, we generate the points of the process sequentially. The thinning procedure occurs at each step of the simulation when a new point is added. If the nearest point is closer than the hard-core distance D, then the new implanted point is deleted.

Hard-core point processes are widely used in practical applications to model repulsion phenomena.

3.3 Marked point processes

A marked point process is a point process for which a characteristic is attached to each point. In mathematical terms, a marked point process on $\mathbb{R}^d$ is a random sequence $\{(x_n, m_n)\}$ where the points x_n constitute a point process (unmarked) called the ground process and the m_n are the marks corresponding to the respective points. A marked point process can also be seen as a point process on $\mathbb{R} \times \mathcal{M}$, where $\mathcal{M}$ is a locally compact space with countable base. This lead to the rigorous definition

Definition 3.3.1 *A marked point process in $\mathbb{R}^d$ with mark space M is a simple point process X in $\mathbb{R}^d \times \mathcal{M}$ with intensity measure θ satisfying $\theta(C \times \mathcal{M}) < \infty$ for all compact set C in $K(R^d)$.*

The marks can be continuous or discrete variables. A marked point process is said to be stationary if its ground process is stationary. Similarly, a marked Poisson point process is simply a marked point process whose ground process is Poisson.

Definition 3.3.2 *The intensity measure of a marked point process X on $\mathbb{R}^d \times \mathcal{M}$ is*

$$\theta(B \times L) = \mathbb{E}(X(B \times L)), \tag{3.20}$$

where B is a Borel set of $\mathbb{R}^d$ and L a measurable set of $\mathcal{M}$.

Intuitively, $\theta(B \times L)$ is the mean number of points in B that have their mark in L. The Campbell formula can be generalized to the case of marked point processes. Hence, let X denote a marked point process on $\mathbb{R}^d \times M$. Then, we have

$$\mathbb{E}\left\{ \sum_{(x,m) \in X} f(x, m) \right\} = \int f(x, m) d\Theta(x, m) \tag{3.21}$$

for any non-negative function f.

It can be shown (see for instance [32]) that the intensity of a marked point process can be decomposed in the following manner

$$d\Theta(x, m) = d\theta(x) dM_x(m), \tag{3.22}$$

where θ is the intensity measure of the ground point process and M_x is a probability measure on $\mathcal{M}$. We interpret M_x as the mark distribution of a point at location x.

For a stationary marked point process, for all subsets L of $\mathcal{M}$, $\Theta(. \times L)$ is a translation-invariant measure, so that, for all Borelien set B in $\mathcal{B}(\mathbb{R}^d)$:

$$\Theta(B \times L) = \theta_L \mu_d(B), \tag{3.23}$$

where μ_d is the Lebesgue measure on $\mathbb{R}$. The quantity θ_L is the intensity of $\mathcal{P}$ with respect to L, and can be interpreted as the mean number of points of $\mathcal{P}$ per unit volume with marks in L. Obviously, if $L = \mathcal{M}$, we have $\theta_L = \theta$, where θ is the intensity of the ground point process.

Example Let X be a Poisson point process in the plane $\mathbb{R}^2$ with intensity θ. To each point $x_n \in X$, we associate a random mark m_n drawn from the uniform law on $[0, 1]$. All marks are drawn independantly. $\{(x_n, m_n)_{x_n \in X}\}$ is a marked point process. The mark space $\mathcal{M}$ is the σ-algebra $([0, 1], \mathcal{B}([0, 1])$.

Problem 3.3.1 *Demonstrate that $\{x_n \in X, m_n > 0.8\}$ is a Poisson point process. What is its intensity?*

The notion of marked point process is fundamental in stochastic geometry and is used in many applications. We will subsequently use marked point processes to study the general Boolean model in chapter 4.

3.4   Additional problems

Problem 3.4.1 *Let X be a stationary Poisson point process. We define the nearest-neighbour distance distribution function Δ to be the distribution of the random distance from a typical point x of X to the nearest other point in the process. Since X is stationary, without lost of generality, it suffices to consider the case where the typical point x is the origin 0. By considering the infinitesimal ball ϵB^d centered around the origin, demonstrate that*

$$\mathbb{P}\{\Delta \leq r\} = 1 - e^{-\theta r^d \mu_d(B^d)},$$

where $\mu_d(B^d)$ is the Lebesgue measure of the unit ball in $\mathbb{R}^d$.

Problem 3.4.2

1. Write a program with Python that simulates a stationary Poisson point process with intensity θ in a bounded squared window.

2. Write a program with Python that simulates a stationary Poisson point process with intensity θ in a disk of radius R.

Chapter 4

Germ-grain processes

4.1 Definition and first properties

Definition 4.1.1 *Let $\Psi = \{x_n; A_n\}$ be a marked point process, where the points x_n lie in $\mathbb{R}^d$ and the marks A_n are random compact subsets of $\mathbb{R}^d$. A germ-grain model can be defined from Ψ by considering the union*

$$A = \cup_{n=1}^{\infty}(A_n \oplus x_n). \tag{4.1}$$

The points x_n are called the germs of the process and the compact sets A_n the grains of the germ-grain model.

In this chapter, we will restrict ourselves to the study of the Boolean model. The Boolean model is an archetypal example of germ-grain process. It is a grain model which is obtained by implanting independant random primary grains A' on the germs $\{x_k\}$ of a Poisson points process $\mathcal{P}$ with intensity θ. Note that primary grains can possibly overlap. The resulting set A is

$$A = \cup_{x_k \in \mathcal{P}} A'_{x_k}, \tag{4.2}$$

where A'_{x_k} denotes the translated of the primary grain A' at point x_k:

$$A'_{x_k} = A' \oplus x_k = \{x_k + y, y \in A'\}. \tag{4.3}$$

Any shape can be used for the primary grains A', including convex, non-convex or even non connected sets. In the literature, A' is commonly referred to as the typical grain of the model.

Definition 4.1.2 *A Boolean model is said to be stationary if the intensity of its germ process is stationary.*

4.1.1   First properties

Lemma 4.1.1 *Let K be a compact set of $\mathbb{R}^n$ and A a Boolean model with primary grain A' and with intensity θ. The number $N(K)$ of primary grains hit by K follows a Poisson distribution of parameter $\mathbb{E}\{\theta(\check{A} \oplus K)\}$:*

$$P\{N = n\} = \frac{\mathbb{E}\{\theta(\check{A}' \oplus K)\}^n}{n!} \exp(-\mathbb{E}\{\theta(\check{A}' \oplus K)\} \qquad (4.4)$$

Proof We denote by $\mathcal{P}$ the germ process associated to A. We can produce a thinned point process $\mathcal{P}_K$ out of $\mathcal{P}$ by deleting all points x_n from $\mathcal{P}$ such that $A'_{x_n} \cap K = \emptyset$. Whether or not a given germ x_n is deleted by this procedure is independant of thinning of other germs. As a consequence, $\mathcal{P}_K$ is an inhomogeneous Poisson point process.

We denote by θ_K the intensity of the thinned process $\mathcal{P}_K$. We have

$$\theta_K(x) = \theta\,\mathbb{P}\{A_x \cap K \neq \emptyset\}.$$

The total number of points of $\mathcal{P}_K$ has a Poisson distribution with mean

$$\bar{N}_K = \theta \int_{\mathbb{R}^d} \mathbb{P}\{A_x \cap K \neq \emptyset\}dx.$$

Since $\mathbb{P}\{A_x \cap K \neq \emptyset\} = \mathbb{P}\{x \in \check{A}_x \oplus K\}$, we have

$$\bar{N}_K = \theta \int_{\mathbb{R}^d} \mathbb{P}\{x \in \check{A}_x \oplus K\}\mathrm{d}x = \theta\mathbb{E}(\mu(\check{A}' \oplus K)),$$

where μ is the Lebesgues measure on $\mathbb{R}^3$. This establishes formula 4.4.

Lemma 4.1.1 guarantees that the number of primary grains in any bounded window remains almost surely finite as long as $\mathbb{E}[\theta(A')] < \infty$. We can easily calculate the Choquet capacity of the boolean model to find

$$T(K) = 1 - \exp(-\mathbb{E}\{\theta(\check{A}' \oplus K)\}) \qquad (4.5)$$

for any compact set K in $\mathcal{K}(E)$. For the stationnary case, the Choquet capacity becomes

$$T(K) = 1 - \exp(-\theta\bar{\mu}(\check{A}' \oplus K)), \qquad (4.6)$$

where $\bar{\mu}(\check{A}' \oplus K)$ denotes the average Lebesgue measure (i.e the average volume) of a primary grain A' dilated by the compact set K.

We can determine the spatial law of the Boolean model by considering the Choquet capacity for the structuring element $\{x\}$. For the stationnary case, according to 4.5, we find

$$q = P\{x \in A^c\} = \exp(-\theta\bar{\mu}(\check{A}')). \tag{4.7}$$

Note that we can easily express the Choquet capacity as a function of q. Hence, we have

$$T(K) = 1 - q^{\dfrac{\bar{\mu}(A' \oplus \check{K})}{\bar{\mu}(A')}}, \tag{4.8}$$

where we have used the relation $A' \oplus \check{K} = -\check{A}' \oplus K$.

Using equation 4.5, it is also possible to calculate the covariance of the Boolean model. Recall that the covariance is defined as a function of vector h of $\mathbb{R}^3$ by

$$C(h) = P\{x \in A, x + h \in A\}. \tag{4.9}$$

$C(h)$ is exactly the Choquet capacity for the structuring element $l_h = \{x\} \cup \{x + h\}$. Thus, we find

$$C(h) = 1 - \exp(-\mathbb{E}\{\theta(\check{A} \oplus l_h)\}). \tag{4.10}$$

For the stationnary case, the covariance yields

$$C(h) = 1 - \exp(-\theta\bar{\mu}(\check{A} \oplus l_h)). \tag{4.11}$$

Proposition 4.1.2 *If A is a Boolean model with typical grain A' and intensity θ, the covariance of A is given by*

$$C(h) = 2p - 1 + (1 - p)^2 \exp(\theta\mathbb{E}(\gamma_{A'}(h))), \tag{4.12}$$

where $\gamma_{A'}(h) = \bar{\mu}(A' \cap A'_{-h})$ is the geometrical covariogram of A' and $p = 1 - q$.

Proof From the probabilistic definition of the covariance, we find

$$
\begin{aligned}
C(h) &= P\{0 \in A \cap A_{-h}\} & (4.13)\\
&= 1 - P\{0 \notin A\} + P\{0 \notin A_{-h}\} - P\{0 \notin A \cup A_{-h}\} & (4.14)\\
&= 2p - 1 + P\{0 \notin A \cup A_{-h}\} & (4.15)
\end{aligned}
$$

In addition, we have

$$P\{0 \notin A \cup A_{-h}\} = \exp(-\theta\bar{\mu}(A \cup A_{-h})) \tag{4.16}$$
$$= (1-p)^2 \exp(-\theta\bar{\mu}(A \cap A_{-h})), \tag{4.17}$$

since $\bar{\mu}(A \cup A_{-h}) = \bar{\mu}(A) + \bar{\mu}(A_{-h}) - \bar{\mu}(A \cap A_{-h})$. This establishes formula (4.12).

Problem 4.1.3 *Demonstrate that the geometrical covariogram of a disk with constant radius R in $\mathbb{R}^2$ is*

$$\gamma(R) = 2R^2 \left(\arccos\left(\frac{h}{2R}\right) - \frac{h}{2R}\sqrt{1 - \left(\frac{h}{2R}\right)^2} \right). \tag{4.18}$$

Problem 4.1.4 *Demonstrate that the geometrical covariogram of a sphere with constant radius R in $\mathbb{R}^3$ is*

$$\gamma(R) = \frac{4\pi R^3}{3}\left(1 - \frac{3h}{4R} + \frac{h^3}{16R^3}\right). \tag{4.19}$$

4.2   Practical implementation

Simulations of random structures are generally performed on a grid of points (i.e 2D or 3D images), using primary grains based on combination of pixels. One can however rely on a completely different approach based upon level sets and implicit functions. In this approach, primary grains are described by implicit functions, which are real valued functions defined in the ambient space. The level sets of an implicit function Φ are described by an equation of the form $\Phi(x, y, z) = c$, for some constant c. A surface is described as a level set of the function Φ, most commonly the set of points for which $\Phi(x, y, z) = 0$. In this case, the points for which $\Phi(x, y, z) < 0$ correspond to the interior of the primary grain associated to the implicit function, the points for which $\Phi(x, y, z) > 0$ to its complementary and the level set $\Phi(x, y, z) = 0$ to the boundary of the primary grain. We can use any primary grain, whatever its shape, as long as we can represent it using an implicit function.

In the implicit function approach, complete simulations are generated using Boolean combinations of primary implicit functions: the union and the
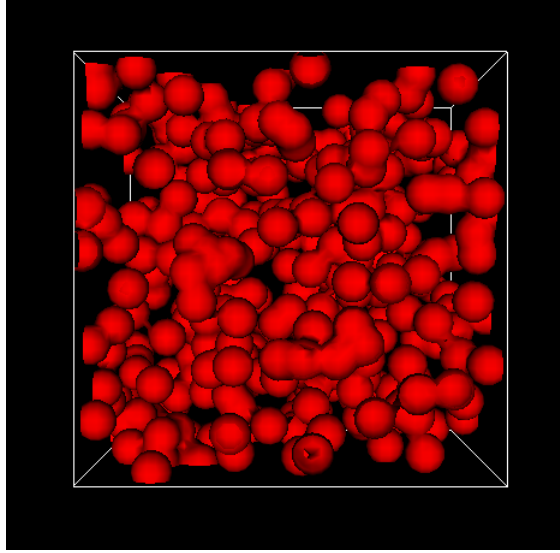
Figure 4.1: 3D Realization of a Boolean model of spheres with constant radius R and intensity θ, realized with the software vtkSim [**?**]. The parameters of the model are $\theta = 5 \times 10^{-2}$ and $R = 1$.

intersection of two objects A_1 and A_2 are defined to yield the minimum and the maximum, respectively, of their corresponding implicit functions. Thus, we have

$$\Phi(A_1 \cup A_2) = \min\{\Phi(A_1), \Phi(A_2)\}$$

and

$$\Phi(A_1 \cap A_2) = \max\{\Phi(A_1), \Phi(A_2)\}.$$

Similarly, the complementary A^c of set A is defined to be the opposite function

$$\Phi(A^c) = -\Phi(A).$$

Overall, using implicit functions to perform the simulation allows us to build complex combinations of simulations that we could not process with a pixel based method. Furthermore, vectorial simulations do not require a large amount of computer resources.

4.3   Statistical analysis for the Boolean model

In this section, we discuss statistical analysis for the Boolean model. For simplicity's sake, we restrict ourselves to the 2D and 3D cases. Our main objective is to determine the parameters of a Boolean model such as its intensity θ or its mean intrinsic volumes.

4.3.1   Method of densities

The method of densities (or intensities) was developed by Weil in 1984 [36] and Santaló [31] to recover the parameters of a Boolean from a given dataset. The main idea of the method is stated in proposition 4.3.1 below.

Proposition 4.3.1 *Let A be some random closed set in $\mathbb{R}^d$. The density of the kth intrinsic volume in $\mathbb{R}^d$ can be estimated from the dataset by relation*

$$v_k = \lim_{r \to \infty} \frac{\mathbb{E}\{V_k(A \cap B(0,r))\}}{\mu_d(B(0,r))}. \tag{4.20}$$

We recall that for $d = 2$, the intrinsic volumes are given by

$$A_A \;\; = \;\; v_2, \tag{4.21}$$
$$L_A \;\; = \;\; 2v_1, \tag{4.22}$$
$$N_A \;\; = \;\; v_0. \tag{4.23}$$
$$\tag{4.24}$$

where A_A and L_A are the mean area and perimeter of the typical grain, respectively. N_A is called specific connectivity number. For $d = 3$, the intrinsic volumes are given by

$$V_V \;\; = \;\; v_3, \tag{4.25}$$
$$S_V \;\; = \;\; 2v_2, \tag{4.26}$$
$$M_V \;\; = \;\; \pi v_1, \tag{4.27}$$
$$N_V \;\; = \;\; v_0, \tag{4.28}$$
$$\tag{4.29}$$

where S_V, M_V and N_V are the surface area, the specific mean curvature and the specific connectivity number, respectively.

For models with convex grains, it is possible to relate the mean values of the intrinsic volumes of the typical grain to measurements conducted on the global dataset through Miles' formulae [24]. For $d = 2$, Miles' formulae yield

$$
\begin{align}
A_A &= p = 1 - \exp(-\theta \bar{A}), \tag{4.30}\\
L_A &= \theta(1 - p)\bar{S} = \theta \exp(-\theta \bar{V}, \tag{4.31}\\
N_A &= \theta(1 - p)\left(1 - \frac{\theta \bar{L}^2}{4\pi}\right). \tag{4.32}
\end{align}
$$

For $d = 3$, Miles' formulae yield

$$
\begin{align}
V_V &= p = 1 - \exp(-\theta \bar{V}), \tag{4.33}\\
S_V &= \theta(1 - p)\bar{S} = \theta \exp(-\theta \bar{V}, \tag{4.34}\\
M_V &= \theta(1 - p)\bar{S}\left(\bar{M} - \frac{\pi^2 \theta \bar{S}^2}{32}\right), \tag{4.35}\\
N_V &= \theta(1 - p)\left(1 - \frac{\theta \bar{M}\bar{S}}{4\pi} + \frac{\pi \theta^2 \bar{S}^3}{384}\right). \tag{4.36}
\end{align}
$$

Hence, when estimates of the densities are given, the intensities of the Boolean model can easily be found. We refer the reader interested by a proof of Miles' formulae to the original paper of Miles [24] and to the book of Schneider and Weil [32].

We can illustrate the methodology for a Boolean model of disks with unknown constant radius R and intensity θ. We suppose that we dispose of a dataset, from which we can estimate a fraction area A_A and a perimeter L_A. The area of the typical grain is constant and yields $\bar{A} = \pi R^2$. Similarly, the perimeter of the typical grain yields $\bar{L} = 2\pi R$. Therefore, Miles' formulae yield

$$
\begin{align}
A_A &= 1 - \exp(-\theta \pi R^2), \tag{4.37}\\
L_A &= \theta(1 - A_A)2\pi R. \tag{4.38}
\end{align}
$$

We can easily solve to find R and θ.

Problem 4.3.2 *We consider a Boolean model of disks of intensity θ whose radius follows a truncated normal distribution centered at value R with variance σ^2. We suppose that we dispose of a dataset, from which we can estimate the fraction area A_A and a perimeter L_A. Determine the parameters of the model.*

4.3.2   Minimum contrast method

The minimum contrast is another statistical method which can be employed to perform parameter identification for germ-grain models. It consists in trying to determine the parameters that minimize the distance between some characteristic function measured on the dataset and the corresponding function obtained either from its theoretical expression or from random realization of the model. For Boolean models, covariance is classicaly used in this purpose, often in combination with granulometry curves. Usually, the graim-germ models are not traceable analytically, and we have to rely on numerical methods to perform the optimization. Nelder-Mead and Levenberg-Marcquart algorithms are often used in this purpose [11].

4.3.3   Stereological mean-value formulae

In many practical situation, we want to study the microstructure of a 3D material through 2D images that correspond to a slice of the material or to a thick section. Hence, we are left with the following question: how to relate the 2D measurements to intrinsic properties of the material?

Let A_v be a spatial stationary grain-germ process in $\mathbb{R}^3$. We assume that the grains are convex. We consider the intersection of A_a of A_v with an arbitrary plane P:

$$A_a = A_v \cap P. \tag{4.39}$$

We denote (x_1, x_2, x_3) an orthonormal system of coordinates in $\mathbb{R}^3$ such that $x_1 \in P$ and $x_2 \in P$. For $r > 0$, we consider the disk

$$c_r = \{x = (x_1, x_2), x_1^2 + x_2^2 < r\}.$$

The number of grains hitting c_r is necessarily the same for A_v and A_a. Hence, using Steiner's formula, we find

$$\theta_v \left(\bar{V} + \frac{\pi \bar{S} r}{4} + \pi \bar{b} r^2 \right) = \theta_a \left(\bar{A} + \bar{L} r + \pi r^2 \right). \tag{4.40}$$

This equation must be valid for all $r > 0$, which proves proposition 4.3.3.

Proposition 4.3.3 *The mean value characteristics of A_v and A_a are related through relations*

$$\theta_v \bar{V} \;=\; \theta_a \bar{A} \tag{4.41}$$

$$\theta_v \frac{\pi \bar{S}}{4} \;=\; \theta_a \bar{L} \tag{4.42}$$

$$\theta_v \bar{b} \;=\; \theta_a. \tag{4.43}$$

Problem 4.3.4 *We consider a material which can be modeled as a Boolean model of spheres with unknown intensity θ and constant radius R in $\mathbb{R}^3$. We suppose that we dispose of experimental images of slices of the material, from which we can estimate a fraction area A_A and a perimeter L_A. Determine the parameters θ and R of the model.*
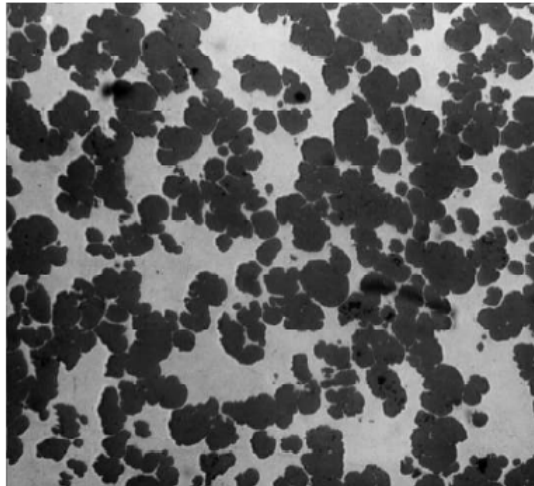

Figure 4.2: Fe-Ag composite microstructure. The microstructure can typically be modeled by a Boolean model of spheres. This illustration is taken from the lecture notes of Jeulin [12]

4.4   Additional problems

Problem 4.4.1 *1. Using Python, implement a code to simulate a random Boolean of sphere in the plane using implicit functions. You can use SMIL library to visualize the result.*
2. Using Python and SMIL library, implement a code to simulate a random Boolean of sphere in the plane, this time by relying on dilation and erosion operators of mathematical morphology.

Problem 4.4.2 *1. The image "boolean2D.png" represents a 2D microstructure that can be modeled by a Boolean model of disks with constant radius. Assuming that the image is ergodic, determine the intensity of the Boolean model and the radius of the spheres.*
2. The image "boolean3D.png" is a 2D slice of a 3D microstructure that can be modeled by a Boolean model of spheres with constant radius. Assuming that the image is ergodic, determine the intensity of the Boolean model and the radius of the spheres.
3. The image "boolean2Dgamma.png" represents a 2D microstructure that can be modeled by a Boolean model of disks whose radius is given by a gamma law with shape parameter k and scale λ. Using the method of contrast and Miles' formulae, determine the parameters of the model.

4.5   Notes

The Boolean model is an archetypal model of stochastic geometry. Reference textbooks on this topic include Matheron [20], Serra [33], Stoyan, Kendall and Mecke [4] and Baccelli and Blaszczyszyn [3]. We also refer the reader to the lecture notes of Jeulin [12]. For an extensive presentation, we refer the reader to the book of Schneider and Weil [32].

Regarding statistical analysis for the Boolean model, we refer the reader to the papers of Weil [36] and Molchanov [25]. Miles' formulae were derived by Miles in 1976 [24]. An introduction can be found in the book of Stoyan, Kendall and Mecke [4]. This topic is discussed more thoroughly in the book of Schneider and Weil [32].

Another topic of interest regarding the Boolean model is percolation. This topic has been studied extensively over the years. We refer the reader interested by this the percolation proerties of the Boolean model to the papers of Hall [9] and Jeulin and Moreaud [17], and to the book of Torquato [35].

In materials engineering, the Boolean model has been employed to simulate a wide range of microstructures. Various examples of the application of the Boolean model in materials science are described in the paper [10] of Hermann. In 1992, Quenec'h *et al.* used the Boolean model to study the growth of WC grains in WC-Co cermets [30]. In 2001, Jeulin et *al.* relied on Poisson polyhedra to simulate the microsctructure of needle-shaped gypsum crystal grains [16]. In 2003, Capasso studied the application of the Boolean model to the description of crystallisation in metals and polymers. More recently, Jean et *al.* simulated the microstructure of rubbers by considering a multiscale Cox-Boolean model [11]. Using a random walk based model, Altendorf and Jeulin developed a stochastic model for simulating 3D fiber structures [1]. Pereyga *et al.* relied on a Boolean model of random cylinders to model a random fibrous network [29].

Chapter 5

Random Tessellations

A *tessellation* or *mosaic* is a division of the $d-$dimensional Euclidean space $\mathbb{R}^d$ into polyhedra. Such geometrical patterns can be observed in many natural situations, as shown in figure 5.1. Hence, random tessellation models have been widely used in physics, materials science and chemistry.

5.1 General introduction

Definition 5.1.1 *A tessellation in $\mathbb{R}^d$ is a countable system $\mathcal{T}$ of subsets satisfying the following conditions:*

- *$\mathcal{T} \in \mathcal{F}_{lf}(\mathbb{R}^d)$, meaning that $\mathcal{T}$ is a locally finite system of nonempty closed sets.*

- *The sets $K \in \mathcal{T}$ are compact, convex and have interior points.*

- *The sets of $\mathcal{T}$ cover the space,*

$$\cup_{K \in \mathcal{T}} K = \mathbb{R}^d \tag{5.1}$$

- *If K and K' are two sets of $\mathcal{T}$ then their interiors are disjoint.*

We denote by $\mathbb{T}$ the set of all tessellations.

The faces of a cell $\mathcal{C}$ of the tessellation are the intersections of $\mathcal{C}$ with its supporting hyperplanes. A $k-$face is a face of dimension k. Among all possible $k-$faces, the $0-$faces, or vertices, and the $1-$faces, or edges, are of particular

interest. The $d-1$ dimensional faces of a $d-$dimensional polytope will be referred to as its facets.
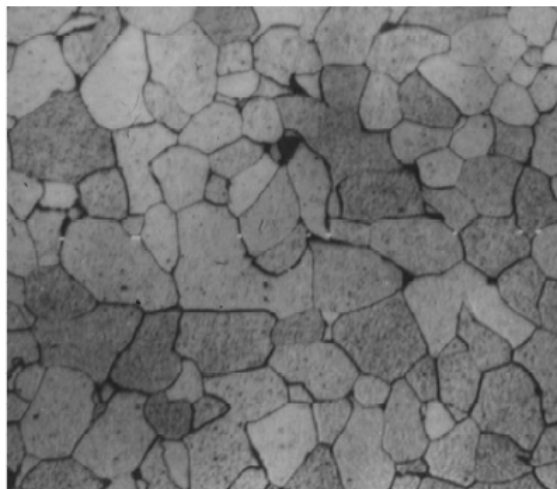
Figure 5.1: Steel polycristal microstructure

Proposition 5.1.1 *The cells of a tessellation $\mathcal{T}$ are convex polytopes.*

Proof Let m be a mosaic and $C \in m$. Since m is locally finite, there are only a finite number of cells, say $C_1, C_2, .., C_m \in m\{C\}$ that intersect C. Since a mosaic covers the whole space $\mathbb{R}^d$, the boundary C of C is found to be

$$C = \cup_{i=1}^{m}(C_i \cap C).$$

By definition, for each i between 0 and m, the relative interiors of C and C_i are disjoint. Therefore, the convex bodies C and C_i can be separated by a hyperplane H_i. More precisely, there exists an hyperplane H_i such that the closed halfspaces H_i^+ and H_i^- bounded by H_i satisfy $C \subset H_i^+$ and $C_i \subset H_i^-$. Note that this is only true because we are considering convex bodies. As a consequence, we have

$$C \subset \cap_{i=1}^{m} H_i^+.$$

Reciprocally, let x be in $\cap_{i=1}^{m} H_i^+$. We suppose that $x \notin C$. Let y be an interior point of C. Necessarily, $y \in \cap_{i=1}^{m} H_i^+$. The line segment with end

points x and y obviously contains a boundary point x' of the cell C. On the one hand, since $x \neq x'$, $x' \in \cap_{i=1}^{m} H_i^+$. On the other hand, $x' \in C_j$ for some $j \in \{1, .., m\}$. This leads to a contradiction. Therefore, if x is in $\cap_{i=1}^{m} H_i^+$, then x is necessarily in C. We have demonstrated that

$$C = \cap_{i=1}^{m} H_i^+.$$

Being compact and the finite intersection of closed halfspace, C is necessarily a convex polytope.

5.1.1 General study

The general study of tessellations is rather technical, and falls beyond the scope of this introductory course. Therefore, in this chapter, we will only try to point out the general ideas behind the theory. We refer the reader interested by a more comprehensive study of general tessellation to the literature.

A fruitful idea to study general tessellations is to rely on the stochastic structures induced by the tessellation on the ambient space. For instance, if we consider a tessellation of the plane $\mathbb{R}^2$, the edges of the tessellation can be seen as a segment process. A d−dimensional tessellation $\mathcal{T}$ also induces point processes in $\mathbb{R}^d$. For instance, the set of vertices of $\mathcal{T}$, the set of edges midpoints or the set of all cells centroids are random point processes of $\mathbb{R}^d$. By determining mean-value formulae for these point processes, it is possible to characterize some of the geometrical properties of $\mathcal{T}$. For instance, the intensity of the random point process constituted by all cells centroid correspond the mean number of cells of the tessellation per volume unit. The number $n_{d-1,d}(x)$ of edges emanating from the vertex x or the number of cells containing x are additional features of interest. Again, mean values for these quantities characterize the geometry of the tessellation. The determination of mean-values formulae for these quantites is a difficult problem, which often builds upon marked point processes theory.

Another common approach is to rely on germ-grain processes theory. Let $\mathcal{T}$ be a tessellation of $\mathbb{R}^d$. If we denote by x_n the cell centers, then $\{(x_n, \mathcal{C}_n)\}$ can be seen as a germ-grain process with convex grains. We call typical cell and we note $\mathcal{C}^0$ the typical grain of the process. In an analogous manner, the edges midpoints with their cooresponding edges form a germ-grain process.

The advantage of this approach is that one can rely on results obtained for germ-grain processes to study a tessellation $\mathcal{T}$.

5.1.2   Random tessellation in the plane

In this section, we try to illustrate the study of random mosaics for a planar tessellation. Thus, let $\mathcal{T}$ be a tessellation on $\mathbb{R}^2$. The following mean values are of particular interest to characterize $\mathcal{T}$:

- θ_k : Intensity of the point process of the centroids of the k-faces induced by $\mathcal{T}$ on $\mathbb{R}^2$.

- $\bar{\mathcal{A}}$: Mean area of the typical cell.

- $\bar{\mathcal{P}}$: Mean perimeter of the typical cell.

- n_{jk} : Mean number of k-faces adjacent to the typical j-faces of $\mathcal{T}$.

Proposition 5.1.2 *The parameters of $\mathcal{T}$ satisfy*

$$\theta_1 = \theta_0 + \theta_2 \tag{5.2}$$

$$n_{02} = 2 + 2\frac{\theta_2}{\theta_0}, \quad n_{20} = 2 + 2\frac{\theta_0}{\theta_2}, \tag{5.3}$$

$$\bar{\mathcal{A}} = \frac{1}{\theta_2}, \quad \bar{\mathcal{P}} = 2\frac{\theta_1}{\theta_2}l_1, \tag{5.4}$$

$$n_{21} = n_{20}, \quad n_{01} = 3, \quad n_{10} = 2. \tag{5.5}$$

In addition, if the tessellation $\mathcal{T}$ is normal, then we have

$$n_{02} = 3, \quad n_{20} = 6. \tag{5.6}$$

These relation are derived by considering the topological configuration of random mosaics. Some results are particularly obvious. For instance, it is clear that the number of neighbor vertices for a given edge is $n_{10} = 2$. Note that similar relations can be obtained in higher dimensions. We refer the reader to the book of Schneider and Weil [32] for a more extensive presentation of the theory and to the book of Stoyan, Kendall and Mecke [4] for the case $d = 3$.

5.2 Poisson tessellation models

5.2.1 Poisson hyperplanes

A hyperplane is a subspace of one dimension less than its ambient space. For instance, if a space is 3-dimensional then its hyperplanes are the 2-dimensional planes. An affine hyperplane is an affine subspace of codimension 1 in an affine space. In Cartesian coordinates, an affine hyperplane H can be described with a single linear equation of the following form

$$u_1 x_1 + u_2 x_2 + \cdots + u_d x_d = r, \tag{5.7}$$

where $\sum_i^d u_d^2 = 1$ and $r \in \mathbb{R}$. The vector $u = (u_1, \cdots, u_d)^T$ is orthogonal to H and unitary. We denote by $\mathcal{A}$ the set of all affine hyperplanes of $\mathbb{R}^d$. An hyperplane is completely characterized by u and r, and can thus be considered as the image of these quantities by the application

$$\Psi : (u, r) \in \frac{1}{2}\mathbb{S} \times \mathbb{R} \to \mathcal{A} \ni H(u, r), \tag{5.8}$$

where

$$H(u, r) = \{x \in \mathbb{R}, u_1 x_1 + u_2 x_2 + \cdots + u_d x_d = r\}. \tag{5.9}$$

and $\mathbb{S}$ is the unit semi-sphere of $\mathbb{R}^d$.

Definition 5.2.1 *Let $\mathcal{P}$ be a Poisson point process in $\frac{1}{2}\mathbb{S} \times \mathbb{R}$ with intensity $\theta(d\mathbf{u})dx$, where θ is a positive Radom measure on the semi-sphere $\frac{1}{2}\mathbb{S}$. The image of $\mathcal{P}$ by application Ψ is the random closed set $\mathcal{H}$ called Poisson hyperplanes network.*

Remark One could have thought of relying on a classical Boolean model with lines as grains to construct Poisson hyperplanes. The problem with this approach is that lines are not bounded and therefore not compact.

Theorem 5.2.1 *Let K be a compact set of $\mathbb{R}^d$. The number of hyperplanes hit by K is a Poisson random variable with intensity*

$$\theta(K) = \int_{\frac{1}{2}\mathbb{S}} \nu_1(K|_{\mathbf{u}})\theta(d\mathbf{u}). \tag{5.10}$$

In this expression, $\nu_1(K|_{\mathbf{u}})$ denotes the total length of the orthogonal projection of K on direction $\mathbf{u}$.

Proof By construction, the intersection of $\mathcal{T}$ with every line with unit support vector $\mathbf{u}$ is a Poisson point process with intensity $\theta(d\mathbf{u})$. Hence, the number of hyperplanes hit by K for a given direction $\mathbf{u}$ is $\nu_1(K|_{\mathbf{u}})\theta(d\mathbf{u})$.

Using theorem 5.2.1, we can easily prove the following proposition.

Proposition 5.2.2 *The Choquet capacity of a Poisson hyperplanes network H is given for all compact sets K in $\mathbb{R}^d$ by*

$$T(K) = 1 - \exp\left[-\int_{\frac{1}{2}\mathbb{S}} \nu_1(K|_{\mathbf{u}})\theta(d\mathbf{u})\right]. \tag{5.11}$$

5.2.2   Poisson lines tessellations

Poisson hyperplanes can be used to produce random tessellations. In this section, we restrict ourselves to the plane $\mathbb{R}^2$.

Definition 5.2.2 *Let $\mathcal{L}$ be a planar motion-invariant line process of intensity θ. $\mathcal{L}$ induces a stationary tessellation on $\mathbb{R}^2$, called Poisson line tessellation. The line intersections form the vertices of the tessellation, and segments of line with vertices at both endpoints form the edges.*

To characterize the tessellation, we introduce the quantity

$$\rho = \frac{2\theta}{\pi}, \tag{5.12}$$

which corresponds to the mean number of lines intersected by a test line segment of unit length. Let g be a fixed line of $\mathcal{L}$. Then, the intensity of the point process of intersection points on g is given by ρ. As a consequence, the mean edge length is

$$l_1 = \frac{1}{\rho} \tag{5.13}$$

With probability one, there are no triplets of lines that meet at the same vertex. Hence, we have

$$n_{02} = 4 \tag{5.14}$$

almost surely. Therefore, using proposition, we find

$$\theta_0 = \frac{\pi\rho^2}{4}, \tag{5.15}$$

$$\theta_1 = \frac{\pi\rho^2}{2}, \tag{5.16}$$

$$\theta_2 = \frac{\pi\rho^2}{4}. \tag{5.17}$$

Definition 5.2.3 *The typical cell of the Poisson hyperplane tessellation is called Poisson polygon.*

The first moments of Poisson polygon can easily be calculated with proposition, to find

$$\bar{\mathcal{A}} = \frac{4}{\pi\rho^2}, \tag{5.18}$$
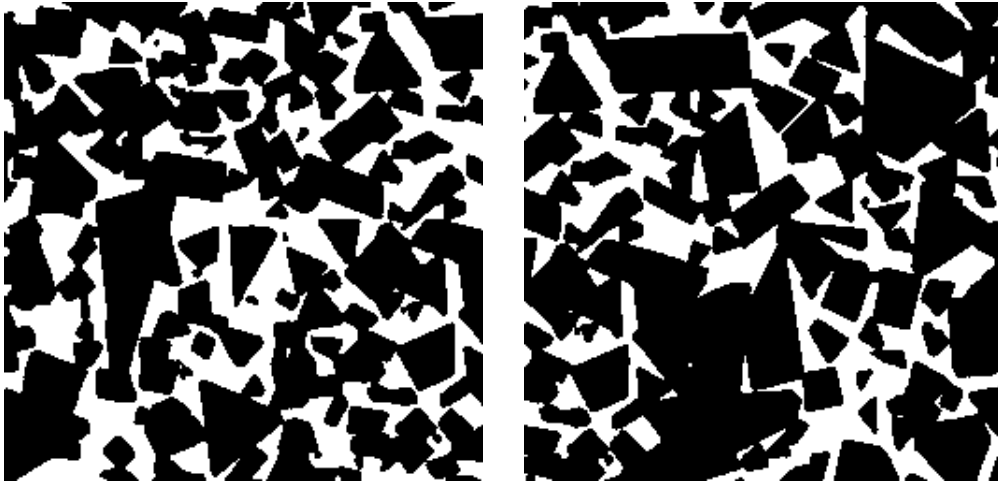
$$\bar{\mathcal{L}} = \frac{4}{\rho}. \tag{5.19}$$

Figure 5.2: Simulations of WC-Co cermets microstructures using Poisson polygons. This figure is taken from the study [30] of Quenec'h *et al.*

5.3   Poisson-Voronoï tessellations

We present in this section the Poisson-Voronoï tessellation model. This model has been studied extensively and is a classical model in stochastic geometry.

5.3.1   Definition

Let Ω denote a given volume in $\mathbb{R}^3$. A Voronoï tessellation is a tessellation built from a Poisson point process $\mathcal{P}$ in the space $\mathbb{R}^3$. Every point x of $\mathbb{R}^3$ is associated to the class $\mathcal{C}_i$ containing all points of $\mathbb{R}^3$ closer from the point x_i of $\mathcal{P}$ than from any other point of $\mathcal{P}$. Hence, the classes $C_i, i = 1, .., N$ are defined by

$$C_i = \left\{ y \in \mathbb{R}^3, \forall j \neq i, \|x_i - y\| \leq \|x_j - y\| \right\}. \tag{5.20}$$

It can be shown that with probability one, Voronoï tessellations are normal and face-to-face. Voronoï tessellations are characterized by one single parameter, namely the intensity of the underlying point process. Thus, according to proposition, for a Voronoï tessellation in the plane, we have

$$\theta_2 = \theta, \tag{5.21}$$
$$\theta_0 = 2\theta, \tag{5.22}$$
$$\theta_1 = 3\theta, \tag{5.23}$$

where θ_0, θ_1 and θ_2 denote the intensities of the point processes constituted by the vertices, the edges center and the cell centers, respectively. Similarly, the mean area of a cell of the tessellation is

$$\bar{\mathcal{A}} = \frac{1}{\theta}. \tag{5.24}$$

These relations can be generalized for $d > 2$.

5.3.2   Johnson-Mehl Tessellations

Johnson-Mehl tesselations can be seen as a sequential version of the Voronoï model, where the Poisson points are implanted sequentially with time. All classes grow then isotropically with the same rate, and the growth of crystal boundaries is stopped when they meet. All Poisson points falling in an

Figure 5.3: Voronoï tessellation in $\mathbb{R}^3$. The simulation has been obtained with the software VtkSim [**?**]. The center of the Voronoï cells are represented in grey.
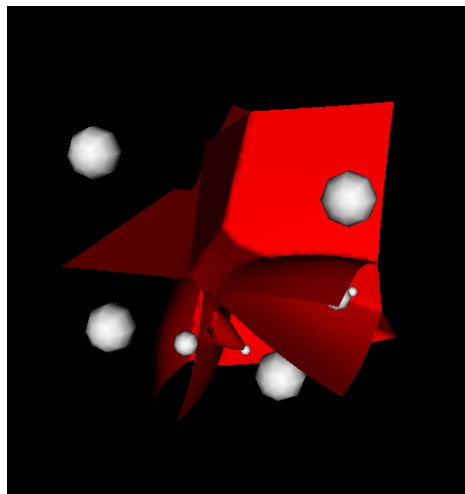

Figure 5.4: Voronoï tessellation in $\mathbb{R}^3$. The simulation has been obtained with the software VtkSim [**?**]. The center of the tessellation cells are represented in grey. The first germs are represented with larger radii.

existing crystal are removed. From a mathematical perspective, a Johnson-Mehl tessellation is constructed from a sequential Poisson point process where the points $x_i, i = 1, .., N$ are implanted sequentially at a time $t_i, i = 1, .., N$. The classes $C_i, i = 1, .., N$ corresponding to the points $x_i, i = 1, .., N$ are defined by

$$C_i = \left\{ y \in \mathbb{R}^3, \forall j \neq i, t_i + \frac{\|x_i - y\|}{v} \leq t_j + \frac{\|x_j - y\|}{v} \right\}. \qquad (5.25)$$

Note that when all times are set to zero, we recover the classical Poisson-Voronoï tessellation model.

5.4   Additional Problems

Problem 5.4.1 *The image "tessellation2D.png" represents a 2D microstructure that can be described by a Voronoï tessellation. Assuming that the image is ergodic, determine the intensity of the Boolean model and the radius of the spheres.*

5.5   Notes

Random tessellations constitute an active topic in stochastic geometry. Reference textbooks on this topic include Matheron [20], Stoyan, Kendall and Mecke [4]. For an extensive presentation, we refer the reader to the book of Schneider and Weil [32], where a proof is given for almost all results. The mean-value relationships given in section 5.1 are mostly due to the studies of Mecke [21] and Møller [26].

Poisson hyperplanes tessellations and Poisson polyhedra have been extensively studied by Matheron [20], Serra [33] and Miles [23]. We also refer the reader to the paper [22] of Mecke.

The Voronoï tessellation is a classical model in stochastic geometry. A general description of Poisson-Voronoï tessellations in $\mathbb{R}^d$ can be found in Møller [26, 28]. The Johnson-Mehl tessellation model was introduced by Johnson and Mehl to describe crystallization processes [18, 2, 8]. Their model

can be seen as a variation of the Voronoï model. The paper [27] of Møller provides a unified exposition of Random Johnson-Mehl tessellations.

Chapter 6

Random functions

6.1 Definition and first properties

In this section, we denote by $\bar{\mathbb{R}} = \mathbb{R} \cup \infty$ the extended real line.

Definition 6.1.1 *A random function on $\mathbb{R}^d$ is a function Z which associates to each element x of $\mathbb{R}^d$ a random variable $Z(x)$ with value in $\bar{\mathbb{R}}$.*

A random function can be characterized by its spatial law, defined for all $m > 0$ by

$$F_m(x, z) = \mathbb{P}\{Z(x_1) < z_1, Z(x_2) < z_2, .., Z(x_m) < z_m\}, \qquad (6.1)$$

where $x_1, x_2, .., x_m$ are points in $\mathbb{R}^d$ and $z_1, z_2, .., z_m$ are elements of $\bar{\mathbb{R}}$. When $m = 1$, the spatial law yields the univariate distribution

$$F(x, z) = \mathbb{P}\{Z(x) < z\}. \qquad (6.2)$$

The function $Z : \mathbb{R} \to \mathbb{R}$ defined by $Z(x) = x + \mathcal{G}(0, \sigma)$, where $\mathcal{G}(0, \sigma)$ is a Gaussian white noise of variance σ^2 is for instance a random function. At each point x, the univariate distribution of Z is given by

$$F(x, z) = \mathbb{P}\{Z(x) < z\} = \mathbb{P}\{\mathcal{G}(0, \sigma) < z - x\}.$$

We define the moments of a random function Z in the following manner:

Definition 6.1.2 *Let Z be a random function on $\mathbb{R}^d$. The covariance of the random function Z at point x is*

$$C(x, h) = \mathbb{E}\{Z(x)Z(x + h)\}. \qquad (6.3)$$

Definition 6.1.3 *Let Z be a random function on $\mathbb{R}^d$. The central correlation function of Z at point x is*

$$W_2(x, x + h) = \mathbb{E}\{Z(x)Z(x + h)\} - \mathbb{E}\{Z(x)\}\mathbb{E}\{Z(x + h)\}. \qquad (6.4)$$

More generally, the central correlation function of order $m > 0$ of Z at point x yields

$$W_m(x_1, .., x_m) = \mathbb{E}\{Z(x_1) - \mathbb{E}\{Z(x_1)\}\}...\mathbb{E}\{Z(x_m) - \mathbb{E}\{Z(x_m)\}\}. \qquad (6.5)$$

6.1.1   Semi-continuity

In most practical situations, the random structures that we study exhibit sharp discontinuities and edges. Hence, the space of continuous functions appear to be too restrictive. In this section, we elaborate on the concept of semi-continuous function, which allows us to describe a wide range of random structures.

Definition 6.1.4 *Let $Z : \mathbb{R}^d \to \bar{\mathbb{R}}$ be a random function. Z is said to be upper semi-continuous at some point x in $\mathbb{R}^d$ if for all $\lambda > 0$, if $\lambda > f(x)$, then there exists a neighboorhood $V(x)$ of x such that $\forall y \in V(x)$, $\lambda > f(y)$. Similarly, Z is said to be lower semi-continuous at some point x in $\mathbb{R}^d$ if for all $\lambda > 0$, if $\lambda < f(x)$, then there exists a neighboorhood $V(x)$ of x such that $\forall y \in V(x)$, $\lambda < f(y)$.*

Obviously, if a function is both lower- and upper-semi continuous, it is continuous.

Semi-continuity and graph   The notion of lower (resp. upper) semi-continuity for a function f is closely related to the topology of the subgraph (resp. overgraph) of f. The subgraph Γ^f of a function f is defined to be the set

$$\Gamma^f = \{(x, z) \in \mathbb{R}^d \times \bar{\mathbb{R}}, z \leq f(x)\}. \qquad (6.6)$$

Similarly, the overgraph Γ_f of a function f is defined to be the set

$$\Gamma_f = \{(x, z) \in \mathbb{R}^d \times \bar{\mathbb{R}}, z \geq f(x)\}. \qquad (6.7)$$
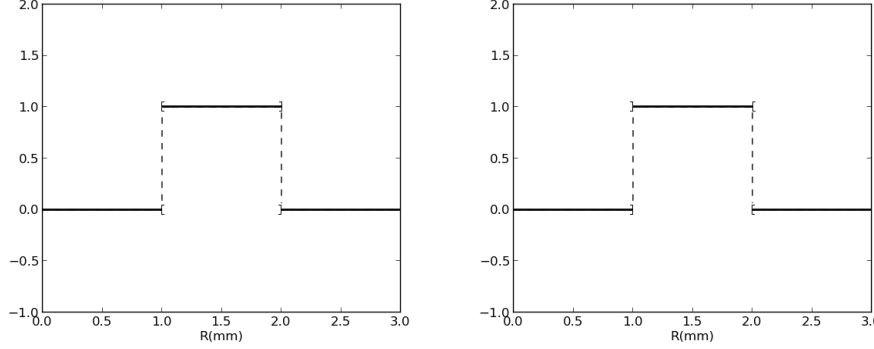
Figure 6.1: Examples of upper (left) and lower (right)semi-continuous functions

Proposition 6.1.1 *Let $Z : \mathbb{R}^d \to \bar{\mathbb{R}}$ be a random function. Z is lower semi-continuous if and only if its overgraph is a closed subset of $\mathbb{R}^d \times \bar{\mathbb{R}}$. Similarly, Z is upper semi-continuous if and only if its subgraph is a closed subset of $\mathbb{R}^d \times \bar{\mathbb{R}}$.*

Proof The proof of proposition 6.1.1 is left as an exercise.

6.1.2 Choquet capacity of a random function

It is possible to generalize the notion of Choquet capacity to the case of semi-continuous functions. To that end, let Z be an upper semi-continuous random function on $\mathbb{R}^d$. We consider the set $\mathcal{L}$ of all lower semi-continuous functions g on $\mathbb{R}^d$ with compact support K. The functions of $\mathcal{L}$ are defined as follows:

$$\begin{aligned} g(x) &< \infty \qquad \text{if } x \in K \\ g(x) &= \infty \qquad \text{otherwise} \end{aligned} \tag{6.8}$$

Definition 6.1.5 *The Choquet capacity of the random function Z is the functional defined on $\mathcal{L}$ by*

$$T_Z(g) = \mathbb{P}\{x \in D_Z(g)\} = 1 - Q(g), \tag{6.9}$$

where

$$D_Z(g)^c = \{x, g(x+y) < Z, \forall y \in K\}. \tag{6.10}$$

As for the case of random sets, the choice of the function g in $\mathcal{L}$ is fundamental and each function brings its own information. For instance, if we consider the function g defined on $\mathbb{R}^d$ by

$$g(x) = \begin{cases} z_i & \text{if } x = x_i \\ \infty & \text{otherwise,} \end{cases} \tag{6.11}$$

then the Choquet capacity yields the univariate distribution of the random function Z.

The Choquet capacity of a lower semi-continuous random function on $\mathbb{R}^d$ can be defined in a similar manner by considering the set $\mathcal{U}$ of all upper semi-continuous functions g on $\mathbb{R}^d$ with compact support K. The functions of $\mathcal{U}$ are defined as follows:

$$\begin{array}{ll} g(x) > -\infty & \text{if } x \in K \\ g(x) = -\infty & \text{otherwise} \end{array} \tag{6.12}$$

Definition 6.1.6 *The Choquet capacity of the lower semi-continuous random function Z is the functional defined on $\mathcal{U}$ by*

$$P_Z(g) = \mathbb{P}\{x \in H_Z(g)\}, \tag{6.13}$$

where

$$H_Z(g)^c = \{x, g(x+y) > Z, \forall y \in K\}. \tag{6.14}$$

Change of support   Let Z be a semi-continuous random function on $\mathbb{R}^d$, and let K be a compact subset of $\mathbb{R}^d$. Then, we define the changes of support by the sup and the inf to be

$$Z_\vee(K) = \sup_{x \in K}\{Z(x) = \vee_{x \in K}\{Z(x)\} \tag{6.15}$$

and

$$Z_\wedge(K) = \inf_{x \in K}\{Z(x) = \wedge_{x \in K}\{Z(x)\} \tag{6.16}$$

Problem 6.1.2 *Let Z be an upper semi-continuous random function on $\mathbb{R}^d$. We consider the lower semi-continuous function g defined on $\mathbb{R}^d$ by*

$$g(x) = \begin{cases} z & \text{if } x \in K \\ \infty & \text{otherwise,} \end{cases} \tag{6.17}$$

Show that

$$T_Z(g) = 1 - \mathbb{P}\{Z_\vee(K)\} \tag{6.18}$$

6.2 Boolean random functions

Boolean random function are an essential extension of the classical Boolean model. We discuss their main properties in this section.

6.2.1 Construction and definition

We denote by μ_d the Lebesgue measure in $\mathbb{R}^d$. Let θ be a σ-finite measure on $\mathbb{R}$, and $\mathcal{P}$ be a stationary Poisson point process with intensity $\mu_d \otimes \theta$. We consider a family of independent lower semi-continuous primary random functions $Z_t'(x)$ such that the subgraphs $\Gamma^{Z_t'} = A'(t)$ have almost surely compact sections $A_{Z_t'}(z)$.

Definition 6.2.1 *We call Boolean random function with primary function $Z_t'(x)$ and intensity $\mu_d \otimes M$ the random function*

$$Z(x) = \sup_{(t_k, x_k) \in \mathcal{P}} \{Z_{t_k}'(x - x_k)\} = \vee_{(t_k, x_k) \in \mathcal{P}} \{Z_{t_k}'(x - x_k)\}. \tag{6.19}$$

Intuitively, we construct a Boolean random function by implanting primary grains at each point of a random point process.

It is sometimes convenient to consider the subgraph of a Boolean random function Z, defined to be

$$\Gamma^Z = \vee_{(t_k, x_k) \in \mathcal{P}} A_{x_k}'(t_k) \tag{6.20}$$

The definition of Boolean random functions can be easily extended to non-stationary Point processes by considering a non-stationary ground point
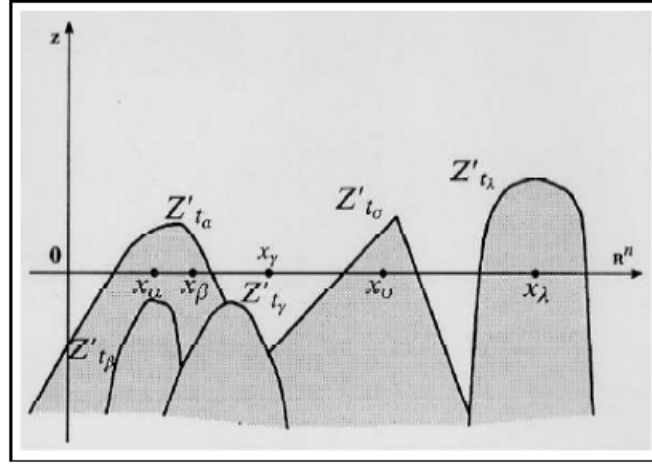
Figure 6.2: Construction of a Boolean random function. At each point of a stochastic point process, we implant a lower semi-continuous primary function Z'_t with random paramater t. This figure is reproduced from the lecture notes of Jeulin [12]

process for $\mathcal{P}$.

An example of Boolean random function is given by the so-called Boolean islands. Let $\mathcal{P}$ be a Poisson point process on $\mathbb{R}^d$. Boolean islands are constructed by implanting at each point of the process the function

$$f(x) = \lambda \delta(x), \tag{6.21}$$

where λ is a random variable on $\mathbb{R}$.

6.2.2   Choquet capacity of a Boolean random function

Let g be an upper semi-continuous function on $\mathbb{R}^d$ with compact support K in $\mathbb{R}^d$. We denote D_z the subset of $\mathbb{R}^d$ defined by

$$D_z(g)^c = \{x, Z(x + y) < g(y), \forall y \in K\}. \tag{6.22}$$

Problem 6.2.1 *Let Z_1 and Z_2 be two lower semi-continuous functions on $\mathbb{R}^d$, show that*

$$D_{Z_1 \vee Z_2}(g) = D_{Z_1}(g) \cup D_{Z_2}(g) \tag{6.23}$$

Theorem 6.2.2 *Let Z be a Boolean random function with primary function $Z'_t(x)$ and intensity $\mu_d \otimes M$:*

$$Z(x) = \vee_{(t_k, x_k) \in \mathcal{P}} \{Z'_{t_k}(x - x_k)\}. \tag{6.24}$$

The Choquet capacity of the Boolean Random function Z is given by

$$1 - T(g) = Q(g) = \exp\left(-\int_{\mathbb{R}} \bar{\mu}(D_{Z'_t}(g)) M(dt)\right) \tag{6.25}$$

Proof According to (6.23), we have

$$D_Z(g) = \cup_{(t_k, x_k) \in \mathcal{P}} D_{Z'_{t_k}}(g). \tag{6.26}$$

By construction, $D_z(g)$ is thus a Boolean model with primary grain $D_{Z'_t}(g)$. Therefore, the number of primary functions Z'_t that intersect the compact support K of g follows a Poisson distribution with parameter $\int_{\mathbb{R}} \bar{\mu}_d(D_{Z't}(g)) M(dt)$. This concludes the proof.

Knowing the Choquet capacity, we can easily calculate the spatial law of the Boolean random function Z by considering the function g defined by

$$g(x) = z_i \text{ if } x = x_i, +\infty \text{ otherwise.}$$

We find

$$1 - T(g) = \mathbb{P}\{Z(x_1) < z_1, .., Z(x_n) < z_n\} =$$
$$\exp\left(-\int_{\mathbb{R}} \bar{\mu}_d(A_{Z'_t}(z_1) \cup ... \cup A_{Z'_t}(z_n)) M(dt)\right). \tag{6.27}$$

In particular, the univariate distribution function is given at any point x in $\mathbb{R}$ by

$$F(z) = \mathbb{P}\{Z(x) < z\} = \exp\left(-\int_{\mathbb{R}} \bar{\mu}_d(A_{Z'_t}(z)) M(dt)\right). \tag{6.28}$$

6.3   Notes

Boolean random function have been introduced by Jeulin and Jeulin [14] for studying rugosity profiles and were investigated in more depth by Serra [33]. An extensive presentation of this topic can be found in the book of Serra [33] and in the lecture notes of Jeulin [12].

In material engineering, random boolean functions have been used to simulate rough surfaces. We refer the reader interested by this topic to the papers of Laurenge and Jeulin [15], Jeulin and Jeulin [14] and Jeulin [13].

Part II

Appendix

Appendix A

Topology of random closed sets

A.1 Closed convergence topology

In this section, we consider a locally compact space of countable type E. Recall that a topological space E is compact if and only if there exists some countable collection $\mathcal{U} = \{U_i\}_{i=1}^{\infty}$ of open subsets of E such that any open subset of E can be written as a union of elements of some subfamily of $\mathcal{U}$. We denote by $\mathcal{F}(E)$, $\mathcal{G}(E)$ and $\mathcal{K}(E)$ the classes of closed, open and compact subsets of E respectively. Similarly, we denote by $\mathcal{P}(E)$ the set of all parts of E.

A.1.1 Closed convergence topology on $\mathcal{F}(E)$

Definition A.1.1 *If B is a subset of E, we denote by $\mathcal{F}_B$ the class of all closed subsets of E intersecting B, and by $\mathcal{F}^B$ its complementary in $\mathcal{F}(E)$.*

$$\mathcal{F}_B = \{F \in \mathcal{F}, F \cap B \neq \emptyset\}, \qquad \mathcal{F}^B = \{F \in \mathcal{F}, F \cap B = \emptyset\} \qquad \text{(A.1)}$$

$\mathcal{F}^B$ is the class of all closed subsets of E disjoint from the subset B.

We can easily show (Problem A.1.1) that the classes $\{\mathcal{F}^K, K \in \mathcal{K}(E)\}$ and $\{\mathcal{F}_G, G \in \mathcal{G}(E)\}$ are stable by union and by finite intersection. In addition, the empty set and E belong to both classes. An immediate consequence is that $\{\mathcal{F}^K, K \in \mathcal{K}\}$ and $\{\mathcal{F}_G, G \in \mathcal{G}\}$ constitute a familly of open sets of $\mathcal{F}$. We denote by $\mathcal{T}_f$ the *closed convergence topology* induced on $\mathcal{F}$ by these classes. Note that the class of subsets

$$\{\mathcal{F}^K_{G_1, G_2, ..G_n} = \mathcal{F}^K \cap \mathcal{F}_{G_1} \cap ... \cap \mathcal{F}_{G_n}, K \in \mathcal{K}, G_1, .., G_n \in \mathcal{G}\} \qquad \text{(A.2)}$$

71

is a basis of the closed convergence topology $\mathcal{T}_f$.

Problem A.1.1 *If $(B_i)_{i \in I}$ is a familly of subsets of E, show that we have*

$$\cup_{i \in I} \mathcal{F}_{B_i} = \mathcal{F}_{\cup_{i \in I} B_i}, \qquad \cap_{i \in I} \mathcal{F}^{B_i} = \mathcal{F}^{\cup_{i \in I} B_i}, \tag{A.3}$$

but only the inclusions

$$\mathcal{F}_{\cap_{i \in I} B_i} \subset \cap_{i \in I} \mathcal{F}_{B_i}, \qquad \cup_{i \in I} \mathcal{F}^{B_i} \subset \mathcal{F}^{\cap_{i \in I} B_i}. \tag{A.4}$$

Theorem A.1.2 *$\mathcal{F}(E)$ is compact and countable for the closed convergence topology.*

Proof The proof proceeds in three steps. First, we construct a countable topological basis of $\mathcal{T}_f$. Then, we show that $\mathcal{F}(E)$ is separated for the closed convergence topology. We finally rely on the topological basis constructed in step one to demonstrate the compacity of $\mathcal{F}(E)$

1/ We first construct a countable topological basis of $\mathcal{T}_f$. Recall that a basis for a topological space equipped with a topology is a collection of open sets for this topology such that every open set can be written as a union of elements of the basis. Let $\mathcal{B}$ be a countable basis of relatively compact open sets for the topology $\mathcal{E}$ of E, such that

$$\forall U \in \mathcal{E}, \quad U = \cup \{ B \in \mathcal{B} | \bar{B} \subset U \}.$$

Let F be an element of $\mathcal{F}(E)$ and $\mathcal{F}^K_{G_1,..,G_n}$ an open neighborhood of F in $\mathcal{F}(E)$. We introduce the family $\mathcal{T}_b$ of subsets of $\mathcal{F}(E)$ defined by

$$\mathcal{T}_b = \{ \mathcal{F}^{\bar{B}'_1 \cup ... \cup \bar{B}'_k}_{B_1,..,B_n} | n, k \leq 0, B_1, .., B_n, B'_1, ..., B'_k \in \mathcal{B} \}.$$

For all i between 1 and n, we can select a point x_i in $F \cap G_i$ and an open set B_i in $\mathcal{B}$ such that $x_i \in \bar{B}_i \subset G_i \cap K^c$. We have built a finite covering of the compact K by a class of open sets $\{ B_j \in \mathcal{B}, j = 1, .., k \}$ such that $\forall j = 1, .., k, \forall i = 1, .., n, \bar{B}_j \cap \bar{B}_i = \emptyset$ and $\bar{B}_j \cap F = \emptyset$. This demonstrates that $\mathcal{T}_b$ is a countable basis of the topology $\mathcal{T}_f$.

2/ The second step of the proof is to show that $\mathcal{F}$ is separated for the closed convergence topology. If F and F' are two distinct closed subsets of

E, there exists $x \in F$ such that $x \in F'$ (or $x \in F'$ such that $x \in F$). Since E is separated, we can find an open set B relatively compact such that $x \in B$ and $F' \cap \bar{B} = \emptyset$. By definition, $F \in \mathcal{F}_B$, $F' \in \mathcal{F}^{\bar{B}}$, and $\mathcal{F}_B \cap \mathcal{F}^{\bar{B}} = \emptyset$. We just exhibited an open set separating F and F' and thus demonstrated that $\mathcal{F}$ is separated.

3/ We finally demonstrate that $\mathcal{F}(E)$ is compact. Let I and J be countable sets, $\{K_i \in \mathcal{K}(E), i \in I\}$ a family of compact sets and $\{G_j \in \mathcal{K}(E), j \in J\}$ a family of open sets, such that

$$\left(\cap_{i \in I} \mathcal{F}_{K_i} \right) \cap \left(\cap_{j \in J} \mathcal{F}^{G_i} \right) = \emptyset. \tag{A.5}$$

If we denote by Ω the union $\cup_{j \in J} G_j$, we have $\cap_{j \in J} \mathcal{F}^{G_j} = \mathcal{F}^{\Omega}$ so that equation A.5 reads

$$\cap_{j \in J} \mathcal{F}_{K_i}^{\Omega} = \emptyset. \tag{A.6}$$

If we assume that for all $i \in I$, $K_i \cap \Omega^c \neq \emptyset$, then, by construction, the closed subset $\overline{(\cup_{i \in I} K_i)} \cap \Omega^c$ is disjoint of Ω and intersects each compact set K_i, which contradicts relation A.6. As a consequence, there exists i_0 in I such that $K_{i_0} \subset \Omega$. Since E is locally compact, we can exhibit a finite covering $G_{j_1}, .., G_{j_n}$ of K_{i_0} by open subsets of E. Moreover, we have

$$\mathcal{F}_{K_{i_0}} \cap \mathcal{F}^{G_{i_1}} \cap .. \cap \mathcal{F}^{G_{in}} = \emptyset. \tag{A.7}$$

This demonstrates that any covering of $\mathcal{F}(E)$ by open subsets of its topological basis contains a finite covering. Thus, $\mathcal{F}(E)$ is quasi-compact. Being also separated, $\mathcal{F}(E)$ is compact.

A.1.2 Closed convergence topology on $\mathcal{G}(E)$

The classes $\{\mathcal{G}^K, K \in \mathcal{K}\}$ and $\{\mathcal{G}_G, G \in \mathcal{G}\}$ constitute a familily of open sets of $\mathcal{G}$. This family induce a topology $\mathcal{T}_g$ on $\mathcal{G}(E)$. The application from $\mathcal{F}$ to $\mathcal{G}$ which associates to each closed set F of $\mathcal{F}$ its complementary F^c in $\mathcal{G}$ is obviously an homeomorphism. As a consequence, we can transpose all topological properties of $\mathcal{F}$ to $\mathcal{G}$ by duality.

A.2   Convergence and continuity in $\mathcal{F}(E)$

A.2.1   Convergence in $\mathcal{F}(E)$

Definition A.2.1 *Let $\{F_n\}_{n\in\mathbb{N}}$ be a sequence of elements of $\mathcal{F}(E)$, and F be a closed subset of E. $\{F_n\}_{n\in\mathbb{N}}$ converges to F if and only if the two following conditions are satisfied:*

1. *If an open set G intersects F, then it intersects all elements F_n of the sequence, except for a finite number of them.*

2. *If a compact set K is disjoint for F, it is disjoint from all elements F_n of the sequence, except for a finite number of them.*

Definition A.2.1 relies on topological considerations, and is quite difficult to use in practice. However, since $\mathcal{F}(E)$ is a countable space, we can restrict our analysis to the case of sequential convergence. Theorem A.2.1 enables us to characterize analytically the convergence in $\mathcal{F}(E)$.

Theorem A.2.1 *Let $\{F_n\}_{n\in\mathbb{N}}$ be a sequence of elements of $\mathcal{F}(E)$, and F be a closed subset of E. $\{F_n\}_{n\in\mathbb{N}}$ converges to F if and only if the two following conditions are satisfied:*

1. *$\forall x \in F$, we can find a sequence $\{x_n\}$ converging to x and $N > 0$ such that, $\forall n > N$, $x_n \in F_n$.*

2. *If a compact set K is disjoint for F, it is disjoint from all elements F_n of the sequence, except for a finite number of them.*

In addition, condition 1 (resp. 2) is equivalent to condition 1 (resp. 2) of definition A.2.1.

Proof Let $\{F_n\}$ be a sequence of elements of $\mathcal{F}(E)$, and F be a closed subset of E.

$1 \rightarrow 1'$: We assume that condition 1 of definition A.2.1 is satisfied: if an open set G intersects F, then it intersects all elements F_n of the sequence, except for a finite number of them. Let x be in F, and $G_1 = E \supset G_2 \supset ...$ be a fundamental system of open neighbourhoods of x. By construction, each open set G_k intersects F. Since condition 1 of definition A.2.1 is satisfied, there exists an integer N_k such that $n \geq N_k$ implies $F_n \cap G_k \neq \emptyset$. As a consequence, we can construct a sequence $\{x_n\}_{n \geq N_1}$, such that $\forall p = N_k, N_k + 1, .., N_{k+1} - 1$, $x_p \in F_p \cap G_k$. $\{x_n\}_{n \geq N_1}$. $\{x_n\}_{n \geq N_1}$ converges to x.

$1' \to 1$: We assume now that condition 1 of theorem A.2.1 is satisfied. Let G be an open set that intersects F. Necessarily, there exists a sequence $\{x_n\}_{n \geq n_0}$ converging to x such that $\forall n \geq n_0, x_n \in F_n$. Since G is an open neighboordood of x, there exists $N \geq n_0$ such that $x_n \in G \cap F_n$. For all $n \geq N$, G intersects F_n.

$2 \to 2'$: If $F = E$, the implication is trivial. If $F \neq E$, let x be a point of and K a compact neighboorhood of x. Since condition 2 of definition A.2.1 is satisfied, there exists $N \geq 0$ such that K is disjoint of F_n for $n \geq N$.

$2' \to 2$: If condition 2 is not satisfied, there exists a compact set K disjoint from F and a subsequence $\{F_{n_k}\}$ such that for any k, $x_{n_k} \in K \cap F_{n_k}$. The subsequence $\{x_{n_k}\}$ has an accumulation point x in $K \cap F^c$. As a consequence, condition 2' is not satisfied.

Problem A.2.2 *Use theorem A.2.1 to show that the application $(F, F') \to F \cup F'$ from $\mathcal{F}(E) \times \mathcal{F}(E)$ to $\mathcal{F}(E)$ is continuous.*

A.2.2 Semi-continuity

Definition A.2.2 *Let $\{F_n\}$ be a sequence of elements of $\mathcal{F}(E)$. We denote by $\underline{\lim} \, F_n$ the intersection of all accumulation points of $\{F_n\}$ in $\mathcal{F}(E)$. Similarly, we denote by $\overline{\lim} \, F_n$ their union.*

Obviously, a sequence $\{F_n\}$ converges in $\mathcal{F}(E)$ if and only if $\underline{\lim} \, F_n = \overline{\lim} \, F_n$.

Proposition A.2.3 *Let $\{F_n\}$ be a sequence of elements of $\mathcal{F}(E)$.*

1. *$\overline{\lim} \, F_n$ is the larger closed set $F \in \mathcal{F}(E)$ satisfying properties 1 of defintion and theorem*

2. *$\underline{\lim} \, F_n$ is closed, and is the smaller closed set $F \in \mathcal{F}(E)$ satisfying properties 2 of definition and theorem*

Proof To prove the first assertion of proposition A.2.3, we consider the set F constituted of all $x \in E$ such that any neighbourhood of x intersects all elements F_n of the sequence $\{F_n\}_{n \geq 0}$ except for a finite number of them. Note that F is a closed set.

On the one hand, if $x \in F$, then x the limit of a sequence $\{x_n\}$ such that $x_n \in F_n$ for n large enough. As a consequence, if x belongs to F, then x belongs to all accumulation sets of the sequence $\{F_n\}_{n \geq 0}$, so that $F \subset \varliminf F_n$.

On the other hand, if xF, we can find a neighbourhood V of x and a partial sequence $\{F_{n_k}\}_{k \geq 0}$ such that $\forall k, V \cap F_{n_k} = \emptyset$. Since $\mathcal{F}(E)$ is a compact space, $\{F_{n_k}\}_{k \geq 0}$ has an accumulation set A such that $\varliminf F_n \subset A$. Since xA, $x\varliminf F_n$. As a consequence, $\varliminf F_n \subset F$.

We conclude this section by defining the notion of semi-continuity for applications in $\mathcal{F}(E)$.

Definition A.2.3 *Let Ω be a topological space, and ψ an application from Ω to $\mathcal{F}(E)$. ψ is upper semi-continuous if for all compact set K in $\mathcal{K}$, the inverse image $\psi^{-1}(\mathcal{F}^K)$ of $\mathcal{F}^K$ is open in Ω. Similarly, ψ is lower semi-continuous if for all open set G in $\mathcal{G}$, the inverse image $\psi^{-1}(\mathcal{F}_G)$ of $\mathcal{F}_G$ is open in Ω*

It is clear that an application is continuous if and only if it is both upper and lower semi-continuous.

Problem A.2.4 *Show that the application $(F, F') \to F \cap F'$ from $\mathcal{F}(E) \times \mathcal{F}(E)$ to $\mathcal{F}(E)$ is lower semi-continuous.*

A.3   Choquet capacity

As pointed out in introduction, statistical approaches provide powerfull methods to study mathematical sets. According to the axiomatic approach of probability theory, the definition of a closed random set in a space E should rely on the construction of a measurable map from some abstract probability space into $\mathcal{F}(E)$.

Problem A.3.1 *Show $\mathcal{F}(E)$ can be equipped with the σ-algebra $\mathcal{B}(\mathcal{F})$ induced by the closed convergence topology, generated by either of the classes*

$$\{\mathcal{F}^K, K \in \mathcal{K}(E)\}, \qquad \{\mathcal{F}_G, G \in \mathcal{G}(E)\}. \tag{A.8}$$

An immediate consequence of problem A.3.1 is that if $\Psi : \Omega \to \mathcal{F}(E)$ is a map from some topological space Ω to $\mathcal{F}$, Ψ is measurable if and only if it is upper or lower semi-continuous. These considerations enable us to define the notion of random closed sets.

Definition A.3.1 *Let $(\Omega, \sigma(\Omega), P)$ be a probability space equipped with its σ-algebra $\sigma(\Omega)$ and a probability measure P. A random closed set A is an $(\sigma(\Omega), \mathcal{B}(\mathcal{F})$-measurable map A from Ω into $\mathcal{F}(E)$. Its distribution is the image measure P_A of P by A.*

Two random closed sets with identical distribution are said to be stochastically equivalent. Similarly, two random closed sets are said to be independant when their joint distribution law is the product of their individual distribution laws.

Definition A.3.2 *Let A be a random closed set of E. The capacity functional T of Z is the functional defined on $\mathcal{K}(E)$ by*

$$T(K) = P_A\{\mathcal{F}_A\} = P\{A \cap K \neq \emptyset\}. \tag{A.9}$$

The distribution of a closed random set is uniquely specified by its capacity functional. Note that for all K in $\mathcal{K}(E)$, $0 \leq T(K) \leq 1$ (i). In addition, if a sequence $\{K_n\}_{n \in \mathbb{N}}$ converges toward K in $\mathcal{K}(E)$, it can be easily proved using that the sequence $T(K_n)_{n \in \mathbb{N}}$ converges toward $T(K)$ (ii). If T is a capacity functional, we can finally consider the functional S_0 on $\mathcal{K}(E)$ defined by

$$S_0(K) = 1 - T(K). \tag{A.10}$$

By recurrence, we define for all $k \geq 1$ a functional S_k on $\mathcal{K}(E)^k$ by

$$S_k(K_0, K_1, .., K_k) = S_{k-1}(K_0, K_1, .., K_{k-1}) - S_{k-1}(K_0 \cup K_k, K_1, .., K_{k-1}). \tag{A.11}$$

Then, for all $K_0, K_1, .., K_k$ in $\mathcal{K}(E)$, $k \geq 0$, $S_k(K_0, .., K_k) \geq 0$ (iii).

Definition A.3.3 *A real function T on $\mathcal{K}(E)$ satisfying properties (i) and (ii) is called a Choquet capacity. A Choquet capacity satisfying property (iii) is said to be alternating of infinite order.*

The main result of this chapter is the Choquet theorem, that we state below without proof. Note that a proof can be found in the books of Matheron [20] or Schneider and Weil [32].

Theorem A.3.2 *If $T : \mathcal{K}(E) \to \mathbb{R}$ is an alternating Choquet capacity of infinite order, then there exists a uniquely determined probability measure P on $\mathcal{F}(E)$ such that, for all compact set K in $\mathcal{K}(E)$,*

$$P\{\mathcal{F}_K\} = T(K). \qquad\qquad (A.12)$$

A.4   Notes

Most of the material of this chapter has been developed by George Matheron. We refer the reader interested by a broader treatment of the topological and stochastic properties of random sets to his treaty [20] published in 1975, and to the more recent book of Schneider and Weil [32]. The Choquet theorem A.3.2 was first established by Choquet [5]. Another relevant reference for the material covered in this chapter is the book of Stoyan, Kendall and Mecke [4].

Appendix B

Basic facts on probability and measure theory

In this appendix, we recall the main results of measure theory and probability theory. A measure on a set is a systematic way to assign a number to each suitable subset of that set. In this sense, a measure is a generalization of the concepts of length, area, and volume. Technically, a measure is a function that assigns a non-negative real number or $+\infty$ to (certain) subsets of a set X. It must assign 0 to the empty set and be countably additive: if we consider a large subset Y of X that we decompose in smaller disjoint subsets, the measure Y will necessarilly be the sum of the measures of the smaller subsets. Probability theory strongly relies on the notion of measure. Probability theory considers measures that assign to the whole set the size 1, and considers measurable subsets to be events whose probability is given by the measure. In this appendix, our aim is to recall some basic facts related to measuretheory and probability.

B.1 σ-algebra

Let Ω be the *fundamental* set of all possible outcomes of a random experiment. The aim of probability theory is to quantify the occurence of some subsets of Ω, called events. We consider for instance all possible outcome of a dice throw. In this case, the fundamental set Ω will be constituted by the outcomes

$$\Omega = \{1, 2, 3, 4, 5, 6\}.$$

Events can be defined as subsets of Ω. For instance, the event "the outcome of the dice throw is 5" simply corresponds to the subset $\{5\}$. This approach allows us to consider more complicated events. For instance, the event "the outcome of the dice throw is NOT 5" corresponds to the subset $\{1,2,3,4,6\} = \{5\}^c$. Similarly, the event "the outcome of the dice throw is strictly less than 5" corresponds to the subset $\{1,2,3,4\}$. In general, we can note than the conjonction or on the contrary the disjonction of events, as well as the negation of events, are events too. In mathematical terms, the set of all events thus verifies the algebraic properties of the σ- algebra.

Definition B.1.1 *A σ-algebra $\mathcal{A}$ on Ω is a class of subsets of Ω such that*
- $\emptyset \in \mathcal{A}$,
- If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$,
- For all countable family $\mathcal{I}$, if for all $i \in \mathcal{I}$, $A_i \in \mathcal{A}$, $\cup_{i \in \mathcal{I}} A \in \mathcal{A}$.
The set Ω along with its σ-algebra $\mathcal{A}$ is called the measurable space $(\Omega, \mathcal{A})$.

Problem B.1.1 *Show that a σ-algebra is stable by intersection.*

Problem B.1.2 *Let Ω be some fundamental set. Check that $(\Omega, \mathcal{P}(\Omega))$ is a measurable space, where $\mathcal{P}(\Omega)$ denotes the set of all subsets of Ω.*

For non countable fundamental sets, the σ-algebra $(\Omega, \mathcal{P}(\Omega))$ can remain highly complicated. Therefore, one often considers simpler σ-algebra generated by some class of subsets of Ω.

Definition B.1.2 *The σ-algebra generated by a class $\mathcal{C}$ of subsets of Ω is the smallest σ-algebra containing $\mathcal{C}$. In particular, when $\Omega = \mathbb{R}^d$, the σ-algebra $\mathcal{B}(\mathbb{R}^d)$ generated by the open sets of $\mathbb{R}^d$ is called the Borelien σ-algebra of $\mathbb{R}^d$.*

B.2   Measures and probability

A measure is simply a functional that associates to each element of a σ-algebra a positive real number. The area in $\mathbb{R}^2$ is a simple example of a measure defined on the measurable space $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$.

Definition B.2.1 *Let $(\Omega, \mathcal{A})$ be a measurable space. A measure on $(\Omega, \mathcal{A})$ is a function $m : \mathcal{A} \to \mathbb{R}_+ \cup \infty$ such that*

- $m(\emptyset) = 0$.
- *m is σ-additive, meaning that*

$$m\left(\cup_{i=1}^{+\infty} A_i \right) = \sum_{i=1}^{+\infty} m(A_i)$$

with $A_i \cap A_j = \emptyset$ if $i \neq j$.

A measure is said to be finite if the measure of the whole space is finite: $m(\Omega) < \infty$. In particular, a probability measure is a measure such that the measure of the fundamental space is $m(\Omega) = 1$. If we go back to our first example of a dice throw, the probability that the results belongs to the set $\Omega = \{1, 2, 3, 4, 5, 6\}$ is indeed 1 and the probability measure of each subset is simply interpreted as the probability of the corresponding event.

Problem B.2.1 *Let P be a probability measure on some measurable space $(\Omega, \mathcal{A})$ and $\{A_i\}$ be some family of events. Show that:*
- *if $A_i \subset A_j$, then $p(A_i) \leq p(A_j)$*
- *$p(\cup_i A_i) \leq \sum_i p(A_i)$.*

An example of measure is provided by the Dirac measure δ_x associated to x:

$$\delta_x(y) = 1 \text{ if } y = x, 0 \text{ otherwise.}$$

Another fundamental example is the indicative function of the subset A of Ω.

$$1_A(x) = 1 \text{ if } x \in A, 0 \text{ otherwise.}$$

B.3 Lebesgue measure

Let $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ be the euclidean measurable space of dimension d with its Borel σ-algebra. A Radon measure on $\mathcal{B}(\mathbb{R}^d)$ is a measure m such that for all bounded subset B of $\mathcal{B}(\mathbb{R}^d)$, $m(B) < \infty$.

Among all Radon measures, Lebesgue measures play a particular role. Lebesgue measures are first defined on hypercubes of $\mathbb{R}^d$ to be

$$\mu(Q) = (x_1^0 - x_0^0)..(x_1^d - x_0^d),$$

where Q is the hypercube $[x_0^0, x_1^0] \times ... \times [x_0^d, x_1^d]$, and can next be generalized to any subset in $\mathcal{B}(\mathbb{R}^d)$. In $\mathbb{R}^3$ (resp. $\mathbb{R}^2$), the Lebesgue measure of a domain is simply its volume (resp. area).

One can easily check that the Lebesgue measure has the property to be isometry-invariant. For instance, in the plane $\mathbb{R}^2$, if we translate and/or rotate some domain, its area remains unchanged. In addition, we have the fundamental result:

Theorem B.3.1 *Let ν be some Radon measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. If ν is isometry-invariant, then there exist a real number $\lambda > 0$ such that $\nu = \lambda \mu_d$, where μ_d is the Lebesgues measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.*

B.4 Measurable functions and random variables

Definition B.4.1 *Let $f : X \to Y$ be some function between two measurable spaces $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$. f is said to be measurable if for all element B of the σ-algebra $\mathcal{Y}$, $f^{-1}(B)$ is an element of the σ-algebra $\mathcal{X}$.*

In practice, most usual functions are measurable. In particular, all continuous functions from $\mathbb{R}^d$ to $\mathbb{R}^{d'}$ are measurable for the Borel σ-algebra.

In probability theory, a random variable is a measurable function from the fundamental set Ω of all possible outcomes of some random experiment. As an example, we consider the events constituted by two dice rollings. The function
$$f : (n_1, n_2) \subset \Omega \times \Omega \to n_1 + n_1 \in \mathbb{N}$$

which associates their sum to the results of two dice rollings is a random variable on $\Omega \times \Omega$.

B.5 Probability density

A direct application of measure theory is the construction of integrals with respect to some measures. A complete exposition of this construction is out of the scope of this appendix, and we refer the readers interested by these

topics to the vast literature on the subject. In this paragraph, we briefly consider the case of probabilities defined by a density functional.

We consider the measurable space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}))$. Let $p : \mathbb{R}^d \to \mathbb{R}$ a non-negative function such that

$$\int_{\mathbb{R}^d} p(r) dr = 1,$$

dr being the Lebesgue measure on $\mathbb{R}^d$. The measure of each Borel set A of $\mathbb{R}^d$ is defined to be

$$P(A) = \int_A p(r) dr.$$

It is clear that the functional $P : \mathcal{B}(\mathbb{R}^d) \to \mathbb{R}_+$ is a probability measure. The function p is the density associated to the probability P and can be interpreted as follows:

$$p(x) dx = P([x, x + dx]).$$

Let f be a random variable on the mesured space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}), P)$. The expectation of f is given by

$$\mathbb{E}[f] = \int_{\mathbb{R}} f(r) p(r) dr.$$

Similarly, its variance is

$$\mathrm{var}[f] = \int_{\mathbb{R}} (f(r) - \mathbb{E}[f])^2 p(r) dr.$$

An example of probability density is provided by the uniform law on some interval $[a, b]$ of $\mathbb{R}$. For the uniform law, the probability density is

$$p(x) = \frac{1}{b - a}.$$

Hence, we find, for all c such that $a \le x \le b$,

$$P\{X > x\} = \int_x^b \frac{dx}{b - a} = \frac{b - x}{b - a}.$$

Another fundamental example of probability density is provided by the gamma distribution. The gamma distribution is characterized by two parameters,

namely the shape parameter k and the scale parameter λ. Its density is given by

$$p(x) = \frac{x^{k-1} \exp(-\frac{x}{\lambda})}{\Gamma(k)\lambda^k}.$$

Problem B.5.1 *Calculate the expectation and the variance of the gamma law.*

B.6 Notes

The lecture notes of Le Gall [19] provide a very good introduction to measure, integration and probability theory.

Bibliography

[1] H. Altendorf and D. Jeulin. Random-walk-based stochastic modeling of three-dimensional fiber systems. *Physical Review E*, 83(4):041804, 2011.

[2] M. Avrami. Kinetics of phase change. I. general theory. *The Journal of Chemical Physics*, 7(12):1103–1112, 1939.

[3] F. Baccelli and B. Blaszczyszyn. *Stochastic Geometry and Wireless Networks: Volume 1: THEORY*, volume 1. Now Publishers Inc, 2009.

[4] S. N. Chiu, D. Stoyan, W. S. Kendall, and J. Mecke. *Stochastic geometry and its applications*. John Wiley & Sons, 2013.

[5] G. Choquet. Theory of capacities. In *Annales de lâinstitut Fourier*, volume 5, page 54, 1953.

[6] M. Coster and J.-L. Chermant. *Précis d'analyse d'images*. Ed. du C. N. R. S., 1985.

[7] M. Faessel. Vtksim software.

[8] E. N. Gilbert. Random subdivisions of space into crystals. *The Annals of mathematical statistics*, pages 958–972, 1962.

[9] P. Hall. *Introduction to the theory of coverage processes*, volume 188. Wiley New York, 1988.

[10] H Hermann. Stochastic models of heterogeneous materials. In *Materials science forum*. Trans Tech Publications, 1991.

[11] A. Jean, D. Jeulin, S. Forest, S. Cantournet, and F. N'Guyen. A multi-scale microstructure model of carbon black distribution in rubber. *Journal of microscopy*, 241(3):243–260, 2011.

[12] D. Jeulin. *Models of random structures*. Lecture notes.

[13] D. Jeulin. Random texture models for material structures. *Statistics and Computing*, 10(2):121–132, 2000.

[14] D. Jeulin and P. Jeulin. Synthesis of rough surfaces by random morphological models. *Stereol. Iugosl*, 3(1):239–246, 1981.

[15] D. Jeulin and P. Laurenge. Simulation of rough surfaces by morphological random functions. *Journal of Electronic Imaging*, 6(1):16–30, 1997.

[16] D. Jeulin, P. Monnaie, and F. Péronnet. Gypsum morphological analysis and modeling. *Cement and Concrete Composites*, 23(2):299–311, 2001.

[17] D. Jeulin and M. Moreaud. Percolation of multi-scale fiber aggregates. In *S4G, 6th International Conference Stereology, Spatial Statistics and Stochastic Geometry. Prague*, pages 26–29, 2006.

[18] W. A. Johnson and R. F. Mehl. Reaction kinetics in processes of nucleation and growth. *Trans. Aime*, 135(8):396–415, 1939.

[19] J.-F. le Gall. Intégration, probabilités et processus aléatoires.

[20] G. Matheron. *Random sets and integral geometry*, volume 1. Wiley New York, 1975.

[21] J. Mecke. Palm methods for stationary random mosaics. *Combinatorial principles in stochastic geometry*, pages 124–132, 1980.

[22] J. Mecke. Inequalities for the anisotropic poisson polytope. *Advances in applied probability*, pages 56–62, 1995.

[23] R. E. Miles. The random division of space. *Advances in applied probability*, pages 243–266, 1972.

[24] R. E. Miles. Estimating aggregate and overall characteristics from thick sections by transmission microscopy. *Journal of Microscopy*, 107(3):227–233, 1976.

[25] I. S. Molchanov. Statistics of the boolean model: from the estimation of means to the estimation of distributions. *Advances in applied probability*, pages 63–86, 1995.

[26] J. Møller. Random tessellations in $\mathbb{R}^d$. *Advances in Applied Probability*, pages 37–73, 1989.

[27] J. Møller. Random johnson-mehl tessellations. *Advances in applied probability*, pages 814–844, 1992.

[28] Jesper Møller. *Lectures on random Voronoi tessellations.* Springer, 1994.

[29] Ch. Peyrega, D. Jeulin, Ch. Delisée, and J. Malvestio. 3d morphological modelling of a random fibrous network. *Image Analysis & Stereology*, 28(3):129–141, 2011.

[30] J.-L. Quenech, M. Coster, J.-L. Chermant, and D. Jeulin. Study of the liquid-phase sintering process by probabilistic models: Application to the coarsening of WC-Co cermets. *Journal of Microscopy*, 168(1):3–14, 1992.

[31] L. A. Santaló. Integral geometry and geometric probability. 1976.

[32] R. Schneider and W. Weil. *Stochastic and integral geometry.* Springer, 2008.

[33] J. Serra. *Image analysis and mathematical morphology.* Academic press, 1982.

[34] P. Soille. *Morphological image analysis: principles and applications.* Springer-Verlag New York, Inc., 2003.

[35] S. Torquato. *Random heterogeneous materials: microstructure and macroscopic properties*, volume 16. Springer, 2002.

[36] W. Weil and J. A. Wieacker. Densities for stationary random sets and point processes. *Advances in applied probability*, pages 324–346, 1984.