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Mathematical Morphology for Boolean Lattices

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2.1 SUMMARY OF BOOLEAN LATTICES

In this chapter we shall add the properties of *distributivity* and *complementation* to the complete lattice. The new structure is called a complete Boolean lattice, or complete Boolean algebra. These new properties, especially the latter, enrich the space directly, with their qualities, and indirectly, by creating a space of points E, underlying the lattice $\mathscr{P}$. This appears clearly in the following review.

Modular lattices, distributive lattices

In any lattice $\mathscr{P}$, for all triplets X, Y, Z, we have

$$X \vee (Y \wedge Z) < (X \vee Y) \wedge (X \vee Z), \tag{2.1}$$

and in particular

$$X < Y \Rightarrow X \vee (Y \wedge Z) < (X \vee Y) \wedge Z. \tag{2.2}$$

The lattice is called *modular* when in (2.2) the second inclusion is an equality, and *distributive* when the more general relation (2.1) is an equality. Any distributive lattice is modular and, by duality, will satisfy the following relation, which corresponds to (2.1):

$$X \wedge (Y \vee Z) = (X \wedge Y) \vee (X \wedge Z).$$

A lattice is distributive if and only if

$$\left.\begin{array}{l} X \wedge Z = Y \wedge Z, \\ X \vee Z = Y \vee Z \end{array}\right\} \Rightarrow X = Y,$$

and modular if and only if this implication holds when X and Y are comparable elements.

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Theorem 2.6 *Let $\{\Gamma_\lambda\}$ be a family of structuring functions on E depending on a parameter $\lambda \geq 0$ such that the class $\Gamma_\lambda(x)$, $\lambda < \lambda_0$, $x \in X$ is inductive for inclusion (i.e. closed under $\uparrow$), and such that for $x \in E$ we have*

$$\dot{\Gamma}_\mu(x)\,\Gamma_{\lambda+\mu}(x) = \Gamma_\lambda(x), \quad \lambda_0 > \{\lambda, \mu\} > 0,$$

the locus of points x_λ such that $\Gamma_\lambda(x_\lambda)$ is maximal in X defines the skeleton *$S(X)$, which is given by*

$$S(X) = \cup \{\cap_{\epsilon > 0} [\dot{\Gamma}_\lambda(X)/\gamma_\epsilon \dot{\Gamma}_\lambda(X)];\ 0 \leqslant \lambda \leqslant \lambda_0\}.$$

As an example, consider the family of Euclidean structuring functions $\Gamma_\lambda : x \to B_\lambda(x)$ that was introduced in Section 2.2. We shall see (Chapter 11) that in $\mathbb{R}^2$ if we take B_λ to be compact discs then the conditions of Theorem 2.6 are satisfied. To digitize a skeleton is not an easy task. Several algorithms have been proposed. A very strong one will be developed in Chapter 13; another, which is more classical, less accurate, and takes longer to implement, is known as *homotopic thinning* (Serra, 1982a, p. 395) (cf. Fig. 2.6).

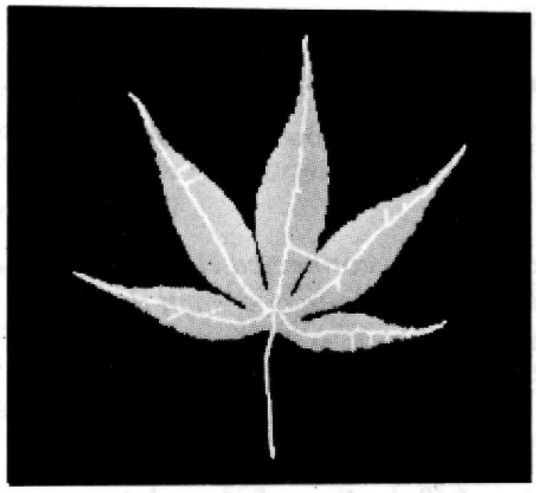


Fig. 2.6 Approximation of a skeleton by homotopic thinning. The least contact between two leaves modifies the homotopy, and since this is preserved in thinning (as well as in the skeleton), long unwanted lines result.

2.6 CONNECTIVITY

The development that we present here is a continuation of a remark by Matheron (1985a), which was itself the development of the concept of the increasing partition in a lattice (Serra, 1984).

Definition 2.7 *We call $\mathscr{C}$ a* connected class *on $\mathscr{P}(E)$ when it is made up of parts of E such that*

(i) $\emptyset \in \mathscr{C}$ *and for all* $x \in E$, $\{x\} \in \mathscr{C}$;
(ii) *for each family* C_i *in* $\mathscr{C}$, $\cap C_i \neq \emptyset$ *implies* $\cup C_i \in \mathscr{C}$.

There is no particular reason why the class $\mathscr{C}$ should be closed under union. However, if we let $\mathscr{C}_x$ denote the subclass of $C \in \mathscr{C}$ that contain a given point x,

$$\mathscr{C}_x = \{C : x \in C \subset \mathscr{C}\},$$

then the union of each non-empty family of elements of $\mathscr{C}_x$ is again in $\mathscr{C}_x$ because of (ii). In other words, the class

$$\mathscr{B}_{\gamma_x} = \mathscr{C}_x \cup \{\emptyset\},$$

closed under union, defines the opening γ_x whose invariant sets constitute $\mathscr{B}_{\gamma_x}$. It is called the *connected opening* of origin x. For all $x \in \mathscr{P}(E)$ we have

$$\gamma_x(X) = \cup \{C : C \in \mathscr{C}, x \in C \subset X\}. \tag{2.20}$$

If $x \notin X$ then $\gamma_x(X)$ is empty. Otherwise we would always have $x \in \gamma_x(X)$ and therefore $\gamma_x(X) \neq \emptyset$, since $\{x\} \in \mathscr{C}$. Thus

$$\gamma_x(X) \neq \emptyset \Leftrightarrow x \in X \Leftrightarrow x \in \gamma_x(X) \in \mathscr{C}.$$

We then say that $\gamma_x(X)$ is the *connected component* of X containing x or marked by x.

Theorem 2.8 *The datum of a connected class* $\mathscr{C}$ *on a Boolean algebra* $\mathscr{P}(E)$ *is equivalent to the family of openings* γ_x *such that*

(iii) *for all* $x \in E$ *we have* $\gamma_x(x) = \{x\}$;
(iv) *for all* $A \subset E$, $x, y \in E$, $\gamma_x(A)$ *and* $\gamma_y(A)$ *are equal or disjoint, i.e.*

$$\gamma_x(A) \cap \gamma_y(A) \neq \emptyset \Rightarrow \gamma_x(A) = \gamma_y(A);$$

(v) *for all* $A \subset E$ *and all* $x \in E$, *we have* $x \notin A \Rightarrow \gamma_x(A) = \emptyset$.

Proof First we show that the datum of $\mathscr{C}$ brings us to the openings γ_x. Axiom (iii) results from $\{x\} \in \mathscr{C}$. To prove (iv), note that $\gamma_x(X) \cap \gamma_y(X) \neq \emptyset$ implies

$$C = \gamma_x(X) \cup \gamma_y(X) \in \mathscr{C}, \qquad \text{with } C \subset X.$$

On the other hand, $\gamma_x(X)$ being non-empty gives

$$x \in \gamma_x(X) \Rightarrow x \in C \Rightarrow C \in \mathscr{C}_x \Rightarrow C \subset \gamma_x(X) \Rightarrow \gamma_y(X) \subset \gamma_x(X).$$

We show the reverse inclusion, and thus equality, in the same way.

Conversely, suppose that we define the class $\mathscr{C}$ as the family of invariant sets of the γ_x, i.e.

$$\mathscr{C} = \{\gamma_x(X), x \in X, X \subset E\}.$$

For $X = \emptyset$ we find $\gamma_x(\emptyset) = \emptyset \in \mathscr{C}$. For $X = \{x\}$ axiom (iii) implies that $\gamma_x(x) = \{x\} \in \mathscr{C}$, and axiom (i) is satisfied. Now let C_i be a family with non-empty intersection in $\mathscr{C}$ and $x \in \cap C_i$. As $C_i \in \mathscr{C}$, we can find a point y_i for each i such that $C_i = \gamma_{y_i}(C_i)$. But $x \in C_i$; therefore, from (iii), $\{x\} = \gamma_x(x) \subset \gamma_x(C_i)$. Thus $\gamma_{y_i}(C_i)$ and $\gamma_x(C_i)$ contain point x, and from (iv) we have $C_i = \gamma_{y_i}(C_i) = \gamma_x(C_i)$. So $\cup C_i = \cup \gamma_x(C_i)$ is invariant under γ_x and belongs to the class $\mathscr{C}$. Thus we have (ii).

We still have to prove that the connected openings associated with this class $\mathscr{C}$ coincide with the γ_x themselves, i.e. to identify the following two classes:

$$\mathscr{C}'_x = \{\gamma_x(A) : \gamma_x(A) \neq \emptyset, A \subset E\},$$
$$\mathscr{C}_x = \{\gamma_y(A) : y \in E, A \in E, \gamma_y(A) \supset \{x\}\}.$$

From axiom (iii) we have $\{x\} \in \mathscr{C}_x$. Let $\gamma_y(A)$ be an element of $\mathscr{C}_x$; then $x \in \gamma_y(A) \subset A$ implies that $x \in \gamma_x(A)$, i.e., from axiom (iv), that $\gamma_y(A) = \gamma_x(A)$. Hence $\mathscr{C}_x \subset \mathscr{C}'_x$. Conversely, set $\gamma_x(A) \in \mathscr{C}'_x$. From axiom (v), we have $\gamma_x(A) \neq \emptyset$; thus $x \in A$ and $\gamma_x(A) \supset \{x\}$, i.e. $\mathscr{C}'_x \subset \mathscr{C}_x$. ■

Corollary 1 *Openings γ_x partition any $A \subset E$ into the smallest possible number of components belonging to the class $\mathscr{C}$, and this partition is increasing in that if $A \subset B$ then any connected component of A is contained in a connected component of B.*

Proof Theorem 2.8 showed that for any $A \subset E$, $x, y \in E$, openings $\gamma_x(A)$ and $\gamma_y(A)$ were either disjoint or identical. We must now show that each point of A belongs to one of the connected openings. To begin with, take the union of $\gamma_x(A)$ when x spans E:

$$\begin{aligned}\bigcup_x \gamma_x(A) &= \bigcup_x \gamma_x\Big(\bigcup_{a\in A} \{a\}\Big) \supset \bigcup_x \bigcup_{a\in A} \gamma_x(a)\\ &= \bigcup_{a\in A} \bigcup_x \gamma_x(a) = \bigcup_{a\in A} \{a\} = A.\end{aligned}$$

We also have the converse inclusion, since $\gamma_x(A) \subset A$ for all x; therefore

$$\bigcup_x \gamma_x = I.$$

Let $A_i = \gamma_x(A)$ be the connected components obtained in this manner. Suppose that we performed another partition of A into elements $A'_j \in \mathscr{C}$. Each $x \in A$ belongs to one A_i and one A'_j; therefore $x \in A_i \cap A'_j$ and $A_i \cup A'_j \in \mathscr{C}$. But since $x \in A_i$ is equivalent to $A_i = \gamma_x(A)$, it follows from (2.20) that $A_i \supset A_i \cup A'_j$ and $A'_j \subset A_i$. The partition of A into the $\gamma_x(A)$ therefore produces the minimum number of connected components. Finally it is increasing because if $x \in A_i = \gamma_x(A)$ then $A \subset B$ implies $x \in \gamma_x(A) \subset \gamma_x(B)$, but $\gamma_x(B)$ is precisely a connected component of B. ■

Corollary 2 *X is connected if and only if for all points y_1 and y_2 of X we can find a connected component Y, included in X, that contains y_1 and y_2.*

Proof Let us fix y_1 and make $y_2 = y$ span X. If for each y_i there exists a Y_i with $\{y_1; y_i\} \subset Y_i \subset X$ then $\bigcap_i Y_i \supset \{y_1\}$. Since $\bigcup_i \{y_1; y_i\} = X$, we have $\bigcup_i Y_i = X$ and X is connected. The converse is trivial. ■

Corollary 3 *For all x, $y \in E$ and all $X \subset E$ we have*

$$y \in \gamma_x(X) \Leftrightarrow \gamma_x(X) = \gamma_y(X) \neq \emptyset,$$

and, in particular,

$$y \in \gamma_x(X) \Leftrightarrow x \in \gamma_y(X).$$

Proof If $y \in \gamma_x(X)$ then we also have $y \in X$; therefore $y \in \gamma_y(X)$. As $\gamma_x(X) \cap \gamma_y(X)$ is non-empty, equality results. Conversely, if $\gamma_x(X) = \gamma_y(X) \neq \emptyset$ then we then have $y \in \gamma_y(X) = \gamma_x(X)$. ■

Examples

(a) The connected class and the partition The definition of a partition, in Section 1.2, seems to resemble that of the connected class. This is confirmed by Corollary 1. In fact, this corollary shows that if we take the set A to be the space E itself then the $\gamma_x(E)$, $x \in E$, partition E.

Conversely, it is clear that any partition $T : E \to \mathscr{P}(E)$ generates a connected class whose connected openings are the γ_x:

$$\gamma_x = I \cap T(x), \qquad x \in E$$

(cf. Fig. 2.7). We can complicate the process by letting the mapping $I \cap T(x)$ operate on each connected component (for a given connectivity) of any set A.

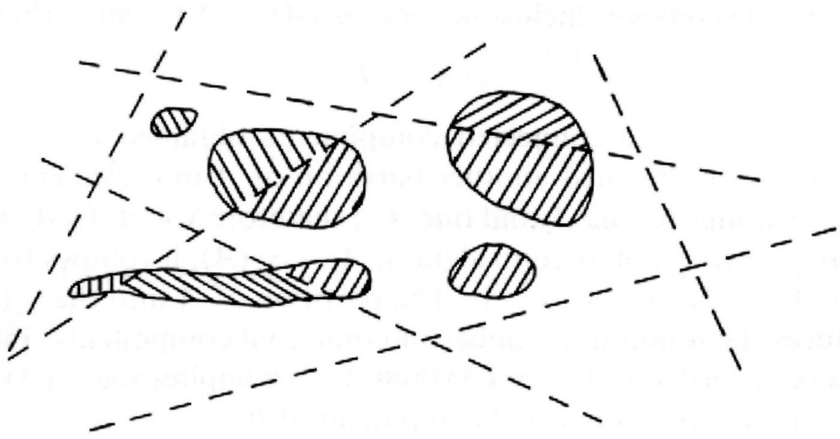
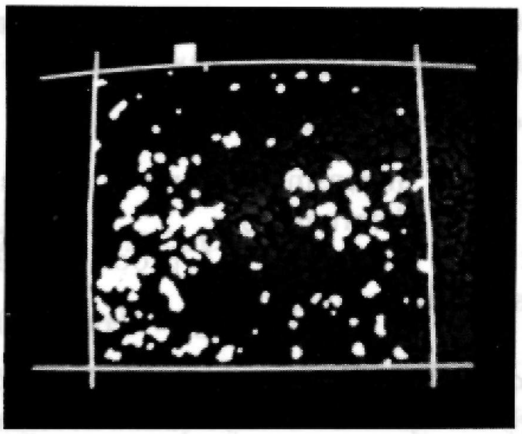
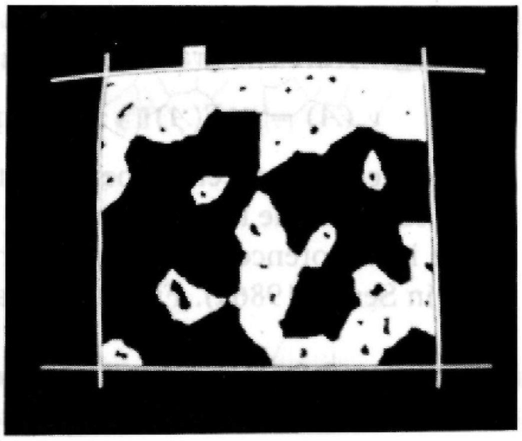


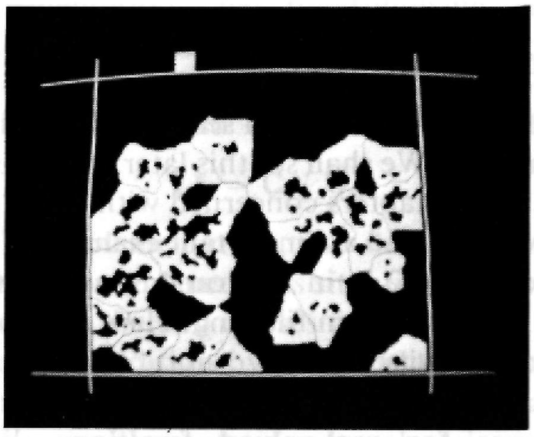
Fig. 2.7 The connected class associated with a partition T.



(a)



(b)



(c)

Fig. 2.8 (a) Porous medium A; (b) Connected components of A invariant under the connectivity $\nu_x = \gamma_x\Gamma \cap I$, where Γ is the hexagonal dilation of size 3. (c) The circled groups of pores considered as new connected components under ν_x (Guedj, 1984).

(b) Extensive dilation and connectivity The principal idea here, which is clearly illustrated in Fig. 2.8, consists in first taking a connected class $\mathscr{C}$ of associated openings γ_x, and then regrouping the connected components of $\mathscr{C}$ that are sufficiently close to one another. This idea generalizes the notion of E-connectivity presented by Guedj (1985). To accomplish it, we use the extensive dilation Γ in the following manner. Let $A \subset E$ be a set of connected components A_i, and let $\gamma_x\Gamma(A)$ be the connected component of $\Gamma(A)$

containing the point x. We shall say that all the A_i that are contained in $\gamma_x\Gamma(A)$ generate *one* grain. This definition is meaningful if the operator

$$(2.21) \qquad \nu_x(A) = \gamma_x\Gamma(A) \cap A \text{ when } x \in A; \ \nu_x(A) = \emptyset \text{ when } x \notin A$$

is effectively a connected opening. It is clearly increasing and anti-extensive, and it satisfies the three axioms (iii) to (v) of Theorem 2.8.

The idempotence of the operator (2.21) is less obvious. A proof of it is given in Serra (1986a). Finally we can state the following.

Proposition 2.9 *Let $\mathscr{C}$ be a connected class on E of associated connected openings γ_x, and let Γ be a structuring function such that for all x we have $x \in \Gamma(x)$, and $\gamma_x\Gamma(x) = \Gamma(x)$. The operator*

$$\nu_x(A) = \gamma_x\Gamma(A) \cap A \text{ when } x \in A; \ \nu_x(A) = \emptyset \text{ when } x \notin A$$

defines a new connected class on E, for which the ν_x are connected openings.

Notions derived from connectivity

Connectivity in the ordinary Euclidean sense, or in digital versions, is often employed in morphology. We shall see this later in reference to some metrics (Section 4.4), and in the chapters concerned with the skeleton (11 and 13 in particular). Moreover, we find an important example of connectivity, in the general sense of Definition 2.7, in Section 4.3, and in Chapter 7, which is entirely devoted to connectivity in filtering. Here are two other notions that are derived from connectivity.

(a) Connected classes for real-valued functions We cannot directly associate connectivity with real-valued functions on $\mathbb{R}^n$ because they are not structured as a Boolean lattice. We can, however, use the function $f: \mathbb{R}^n \to \overline{\mathbb{R}}$ to partition $\mathbb{R}^n$, which permits us to associate connected components with the support of f in $\mathbb{R}^n$, and indirectly with itself. The simplest example of this is to take for partition classes the connected zones of $\mathbb{R}^n$ where $f(x)$ is either $\geq t_0$ or $< t_0$.

(b) Ultimate erosions In practice, the ultimate erosion is often used to count the number of disconnections that appear when a set X is operated on by a sequence of erosions by a family $\dot{\Gamma}_\lambda$ that is decreasing and left-continuous w.r.t. λ. With this goal in mind, we use what is again called the ultimate erosion of X, but taken in a larger sense as Z, the union of the ultimate erosions of all connected components of the family $\{\dot{\Gamma}_\lambda(X), 0 \leq \lambda \leq \lambda_{max}\}$. The set $Z_\lambda(X)$ is one of these ultimate erosions if and only if $Z_\lambda(X)$ is a non-empty connected component of $\dot{\Gamma}_\lambda(X)$ and for all $\mu > \lambda$ we have $\dot{\Gamma}_\lambda[Z_\lambda(X)] = \emptyset$. The ultimate erosion $Z(X)$ is then written

$$Z(X) = \bigcup_{\lambda \geq 0} \bigcap_{\mu > 0} \{\gamma_x(\dot{\Gamma}_\lambda(X));\, x \in \dot{\Gamma}_{\lambda+\mu}(X)\}.$$

2.7 CONCLUSIONS

(a) Levels of generality

The division into two levels—complete lattice, and then Boolean algebra—may seem a bit rough. In $\mathbb{R}^n$, for example, the space of closed sets $\mathscr{F}$, or that of the subgraphs of numerical functions of $\mathbb{R}^{n+1}$, which both play essential roles in Euclidean morphology, are not complemented. They are, however, constructed from the space of points $\mathbb{R}^n$, and duality with respect to complementation is constantly invoked.

This "semicomplementation" has the following cause. Let $\mathscr{P}'$ be a complete sublattice of $\mathscr{P}(E)$, and $\mathscr{P}'^c$ the sublattice, also complete, formed by the complements in $\mathscr{P}(E)$ of the elements of $\mathscr{P}'$. If ψ is a mapping of $\mathscr{P}$ into itself under which both $\mathscr{P}'$ and $\mathscr{P}'^c$ are closed then the mapping

$$\psi^* = \mathbf{C}\,\psi\,\mathbf{C}, \tag{2.21}$$

i.e. the dual of ψ for complementation, also maps $\mathscr{P}'$ and $\mathscr{P}'^c$ (respectively) into themselves. We frequently meet this situation in practice. For example, in Euclidean morphology the two classes of closed and open sets in $\mathbb{R}^n$ are closed under Minkowski addition by a compact set, and the class of subgraphs of u.s.c. numerical functions of $\mathbb{R}^{n-1}$ considered as closed sets of $\mathbb{R}^n$, as well as its dual for complementation, are also closed under Minkowski addition by any compact set of $\mathbb{R}^n$.

Under these circumstances, the results relative to the three dualities (Section 2.2) can be applied without change (see Section 9.4). Nevertheless, we cannot directly transfer the notions that make use of the existence of points, such as the skeleton or connectivity. For example, the complete lattice $\mathscr{G}$ of open sets in $\mathbb{R}^n$ does not contain the points of $\mathbb{R}^n$.

We could have generalized our exposition slightly and studied mappings of a complete lattice $\mathscr{P}_1$ into a complete lattice $\mathscr{P}_2$ (Chapter 1), or of $\mathscr{P}(E_1)$ into $\mathscr{P}(E_2)$ (this chapter). In this case we should have lost the possibility of comparing X with $\Gamma(X)$, and consequently of formulating the concept of extensivity, which is very useful with regard to skeletons and families Γ_λ, amongst others. This having been said, all the results of this chapter and the first remain valid when $\mathscr{P}_1$ is a complete sublattice of $\mathscr{P}_2$ (for example $\mathscr{P}_2 = \mathscr{P}(\mathbb{R}^n)$ and $\mathscr{P}_1$ is the class of open sets). This is by far the most useful case.