Set Connections and Discrete Filtering

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Jean SERRA

Ecole des Mines de Paris

Connectivity in Mathematics

Topological Connectivity **:** Given a topological space E, set $A \subseteq E$ is connected if one cannot partition it into two non empty closed sets.

A Basic Theorem **:**

If $\{A_i\}$ $i \in I$ is a family of connected sets, then

$$\{ \cap A_i \neq \emptyset \} \Rightarrow \{ \cup A_i \text{ connected} \}$$

Arcwise Connectivity (more practical for $E = R^n$) **:** A is **arcwise connected** if there exists, for each pair $a,b \in A$, a continuous mapping ψ such that **$[\alpha, \beta] \in R$ and $f(\alpha) = a$; $f(\beta) = b$**

This second definition is more restrictive. However, for the open sets of R^n, both definitions are **equivalent**.

Criticisms

Is* topological connectivity *adapted to Image Analysis ?

Digital versions of arcwise connectivity are extensively used:

- in 2-D : 4- and 8- connectivities (square), or 6- one (Hexagon);
- in 3-D : 6-, 12-, 26- ones (cube) and 12- one (cube-octaedron).

However :

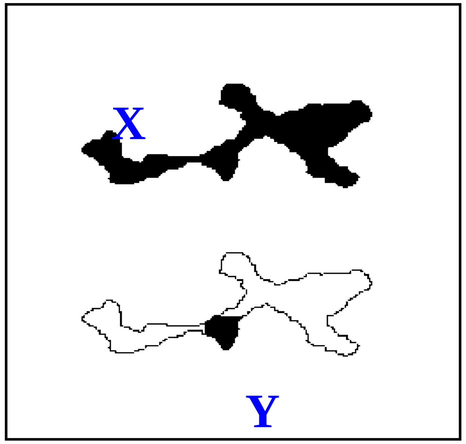
Planar sectioning (3-D objects) as well as sampling (sequences) tend to disconnect objects and trajectories, and topological connectivity does help so much for reconnecting them;

More generally, in Image Analysis, a convenient definition should be operating, *i.e.* should introduce specific operations ;

(Binary) Digital Geodesic Dilation

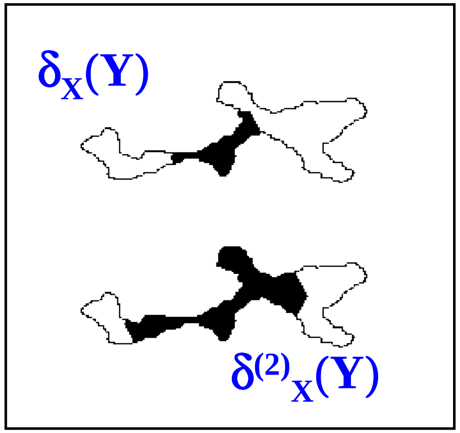
- When E is a digital metric space, and when $\delta(x)$ stands for the unit ball centred at point x, then the unit geodesic dilation is defined by the relation :

$$\delta_X(Y) = \delta(Y) \cap X$$



- The dilation of size n is then obtained by **iteration** :

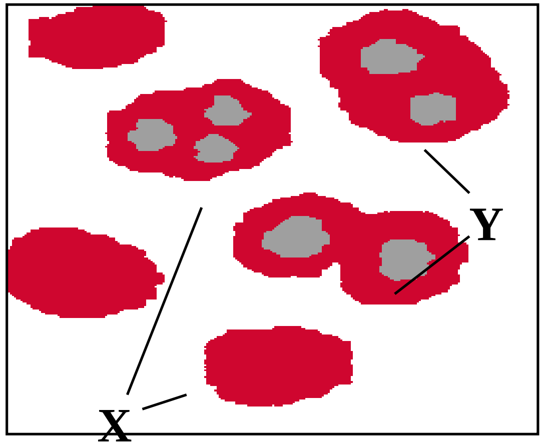
$$\delta^{(n)}{}_X(Y) = \delta(\ldots \delta(\delta(Y) \cap X) \cap X \ldots) \cap X$$



- Note that the geodesic dilations are ***not*** translation invariant.

Binary Reconstruction

- As the grid spacing becomes finer and finer, the digital geodesic dilation tends towards the Euclidean one when X is locally finite union of disjoint compact sets.

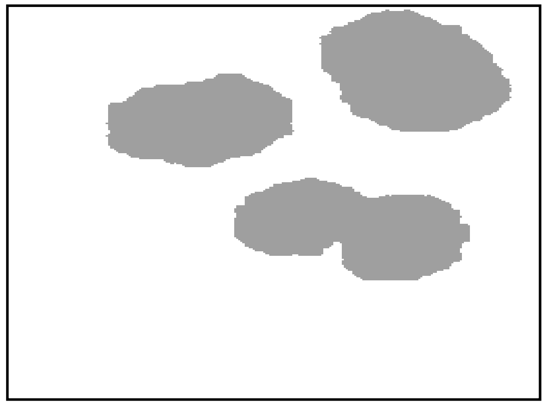


- In such a case, the infinite dilation of Y

$$\delta_{X,\infty}(Y) = \cup\{ \delta_{X,\lambda}(Y) , \lambda>0 \}$$

turns out to be the reconstruction of those connected components of set X that contain at least one point of set Y.

Given Y, it is an ***opening*** of set X.



Connected Opening and Connection

- When marker Y reduces to one point, x say, the reconstruction opening

$$\gamma_x(A) = \cup\ \delta_A^{(n)}(x)$$

called "**point connected opening**", is nothing but the connected component of A that contains point x , or $\varnothing$, if x is outside from A.

- Therefore, as marker x spans space E, it generates a family $\{\gamma_x\}$ of openings that satisfy the three following properties

i) $\gamma_x(x) = \{x\}$ $\quad x \in E$

ii) $\gamma_y(A)$ and $\gamma_z(A)$ $\quad y, z \in E,\ \ A \subseteq E$ **are disjoint or equal**

Binary Connection

- ***Definition :*** Let E be an arbitrary space. A **connection** $\mathcal{C}$(E) on $\mathcal{P}$(E) is a class of $\mathcal{P}$(E) such that

  iv) $\emptyset \in \mathcal{C}$;

  v) $\forall \mathbf{x} \in \mathbf{E} : \{\mathbf{x}\} \in \mathcal{C}$;

  (*class* $\mathcal{C}$ *contains always all points of E plus the empty set*)

  vi) $\forall\{\mathbf{A}\},\ \mathbf{A} \in \mathcal{C} : \quad \{\cap \mathbf{A} \neq \emptyset\} \Rightarrow \{\cup \mathbf{A} \in \mathcal{C}\}$

  (*the union of elements of* $\mathcal{C}$ *whose intersection is not empty is still in* $\mathcal{C}$)

- ***Theorem :*** Every family of openings $\{\gamma_x , x \in E\}$, that satisfy the three above properties generates a **connection**, $\mathcal{C}$(E) on $\mathcal{P}$(E).

  Conversely, every connection $\mathcal{C}$ on $\mathcal{P}$(E) induces a **unique** family of openings satisfying ***i)*** to ***iii)***.

  The elements of $\mathcal{C}$ are the invariant sets of the family $\{\gamma_x , x \in E\}$.

Comments

- This above axiomatic and theorem were proposed in1988 by Matheron and Serra. They had in mind
  - to formalize the *reconstruction* techniques,
  - to make their approach free of any *cumbersome topology of the continuous spaces,*
  - to encompass *more than particles* seen as "one piece objects",
  - to design nice *strong morphological filters.*

But their approach was basically **set wise** oriented. Now, the major use of filtering concern grey tone and colour images (and their sequences):

Can we derive from connected openings pertinent filters for grey images ?

Do we need dilation based reconstruction algorithms ?

Can we express the notion of a connection for Lattices, in general ?

Examples of Connections

- In Digital Imagery, the connected components in the senses of the

  4- and 8-connectivity (***square*** grid) ,

  6-connectivity (***hexagonal*** grid) ,

  12-connectivity (***cube-octahedral*** grid) ,

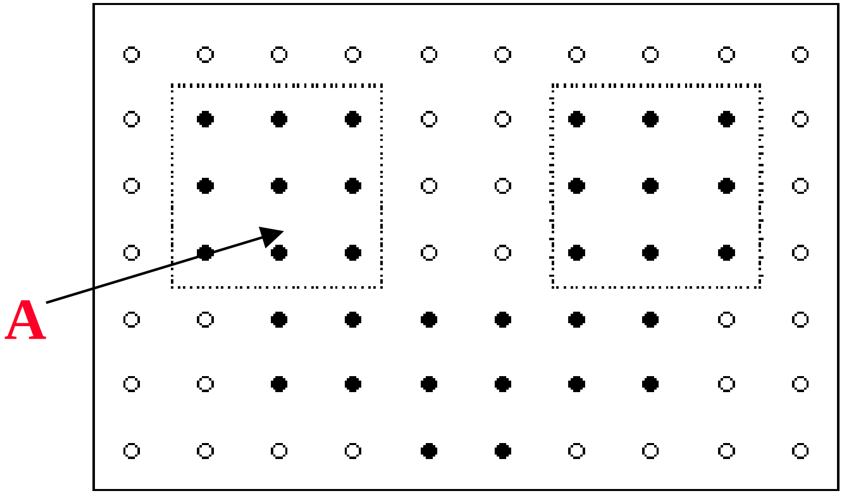
  constitute four different connections.

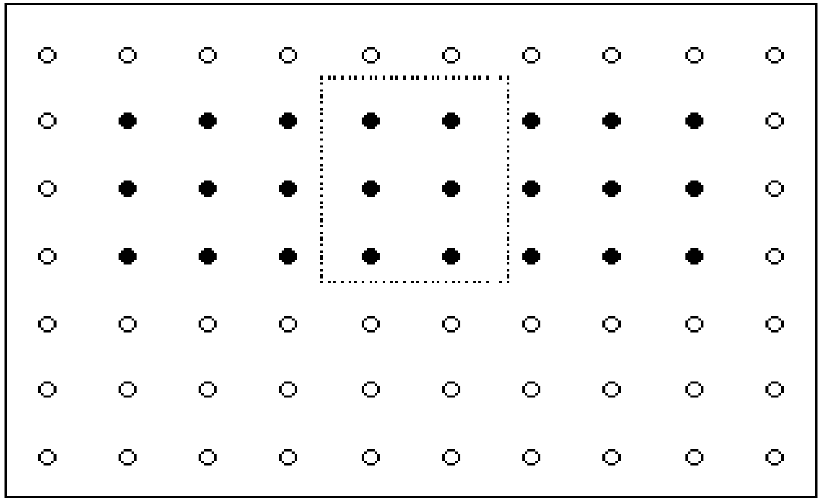
- In Mathematics, both ***topological*** and ***arcwise*** connectivities generate connections.
- The ***second generation connections*** (see below) are new connections that allow to consider clusters of objects as connected entities.
- Also, the notion extends to numerical and to multi-spectral functions.
- Therefore the previous approach gathers under a ***unique axiomatic*** the various usual meanings of "connectivity", plus new ones (*e.g.* the clusters). Note that no topological requirement is needed.

Example of a Connection

The following example is due to Ch. Ronse. Start from a primary connection, and take for class $\mathcal{C}$

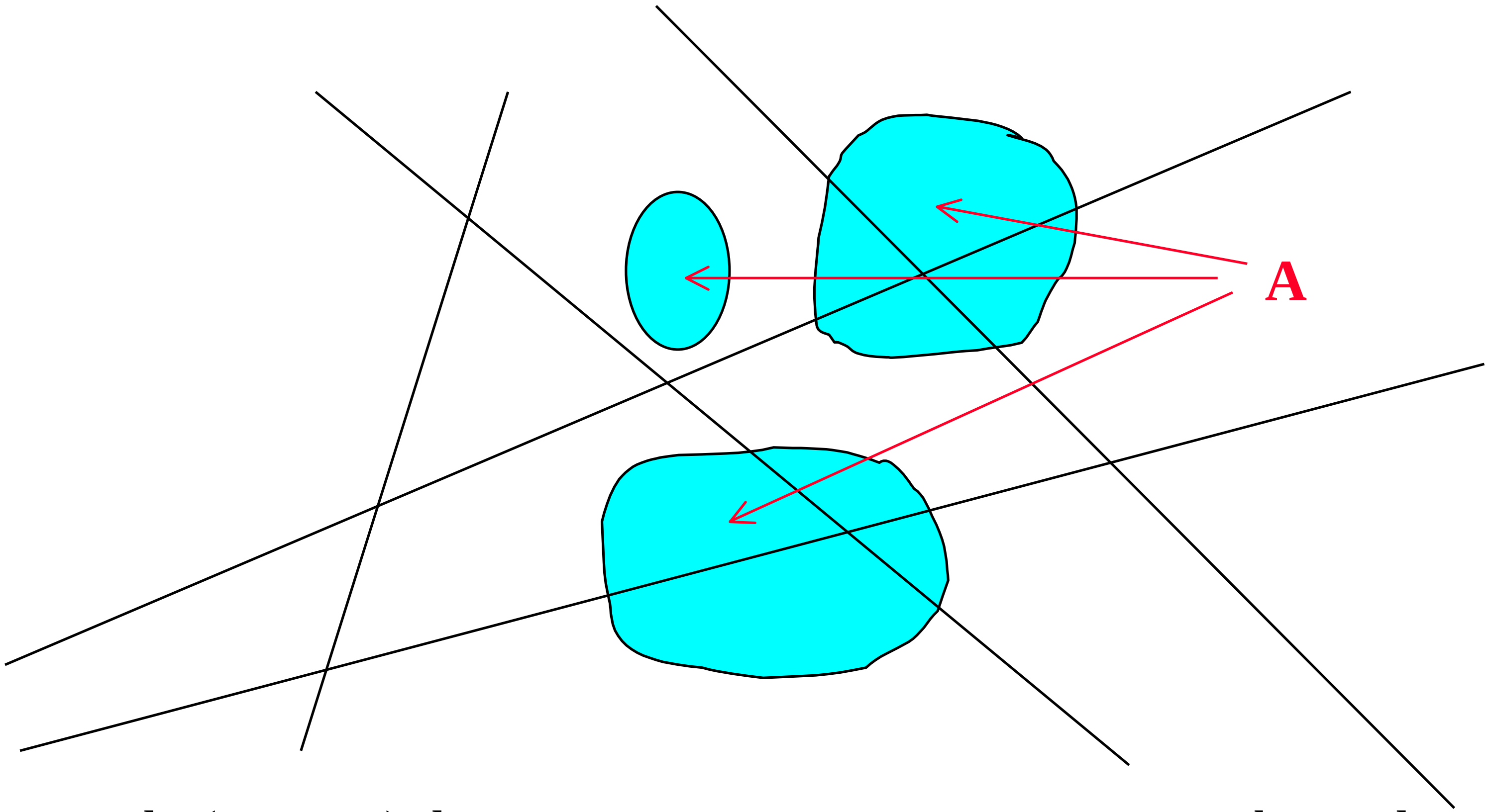
- *all points of E ;*
- *and all connected sets of type that are open by a given B.*





a) set A is made of 14 point components and of two larger ones;
b) under opening-closing, the point pores generate a single component.

Another example of a Connection



- This example (J.Serra) does not require primary connection, but only a given partition of the space, indicated here by the set of lines;
- Then, the connected components of A are the intersection between A and the classes of the partition

Second Generation Connection

Here is another example of a connection (*J.Serra*) which differs from the usual arcwise connectivities.

- ***Proposition1:*** Let $\delta : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ be an extensive dilation that preserves connection $\mathcal{C}$ (*i.e.* $\delta(\mathcal{C}) \subseteq \mathcal{C}$). Then, the inverse $\mathcal{C}' = \delta^{-1}(\mathcal{C})$ of $\mathcal{C}$ turns out to be a **connection** on $\mathcal{P}(E)$, which is **richer** than $\mathcal{C}$.

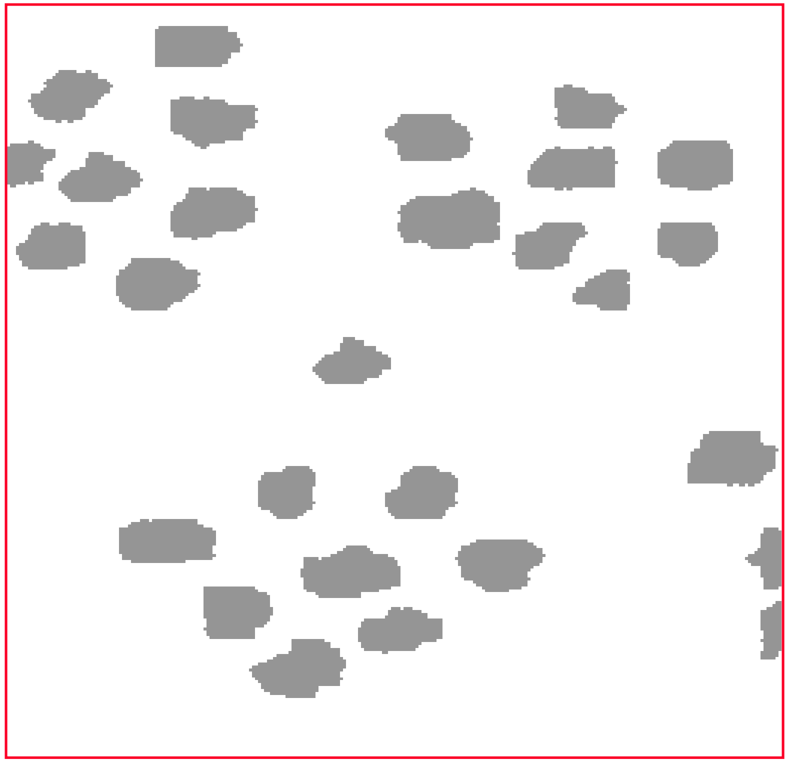
- ***Proposition2* :** The $\mathcal{C}$- components of $\delta(A)$, $A \in \mathcal{P}(E)$, are exactly the **images**, under δ, of the $\mathcal{C}$'- components of A .

If γ_x designates the opening of connection $\mathcal{C}$, and ν_x that of $\mathcal{C}'$, we have:

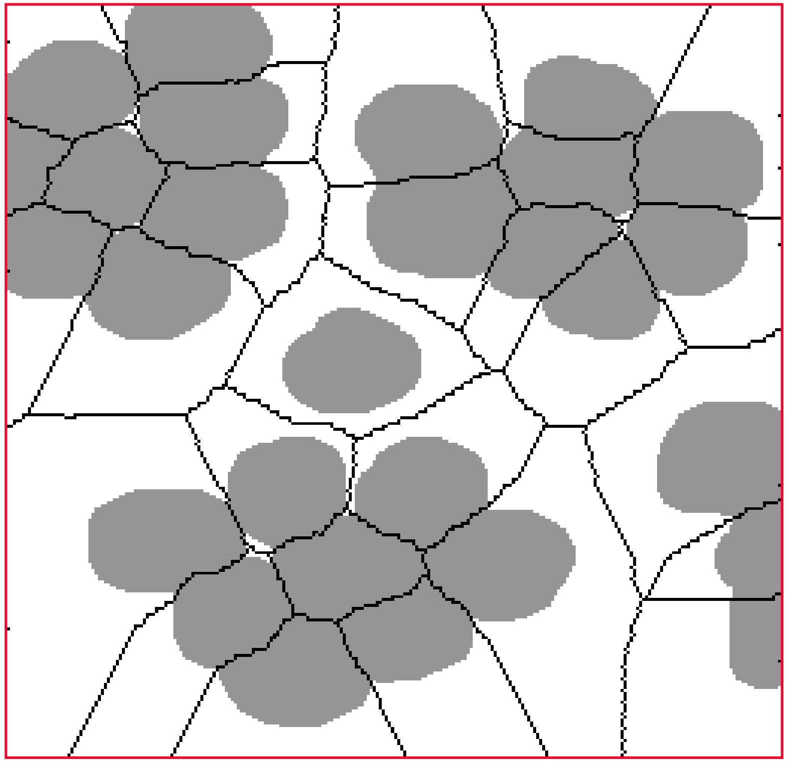
$$\nu_x(A) = \gamma_x \delta(A) \cap A \qquad \text{when } x \subseteq A \quad ;$$
$$\nu_x(A) = \varnothing \qquad \text{when not.}$$

Application : Search for Isolated Objects

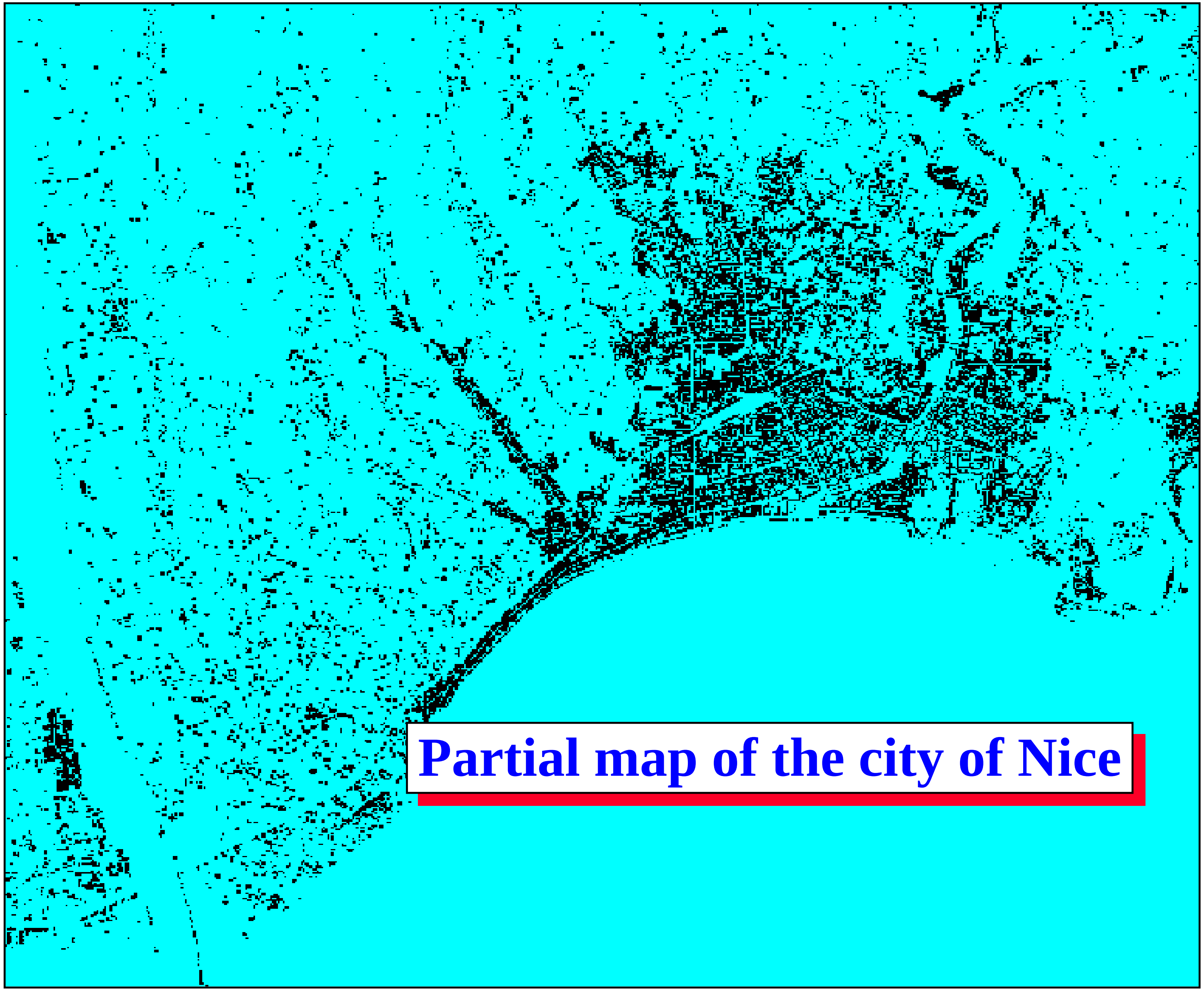
Comment: *One want to find the particles from more than 20 pixels apart. They are the only particles whose dilates of size 10 miss the SKIZ of the initial image.*



a): Initial Image

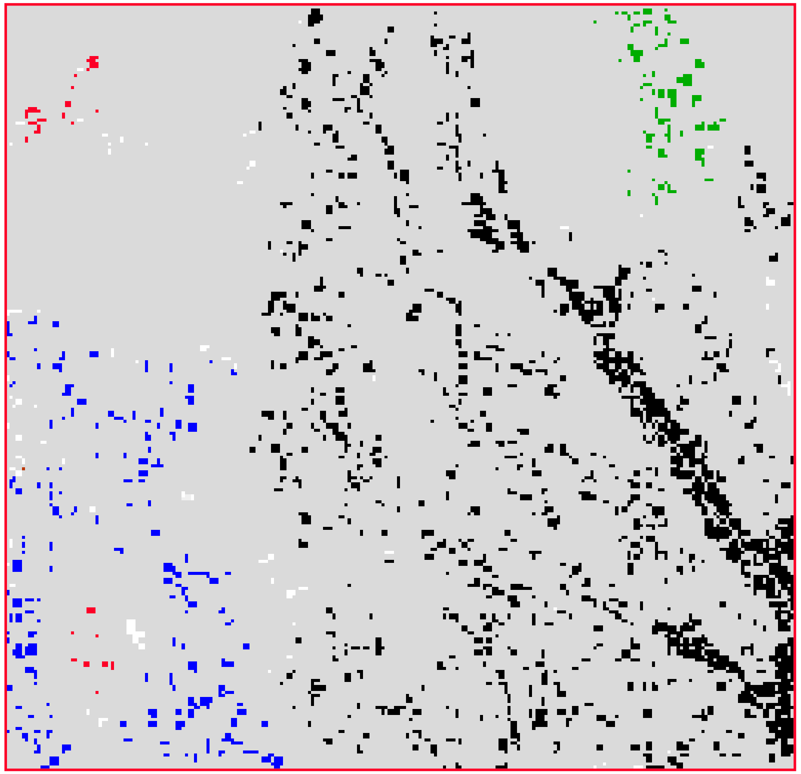


b) : SKIZ and dilate of a) by a disc of radius 10.

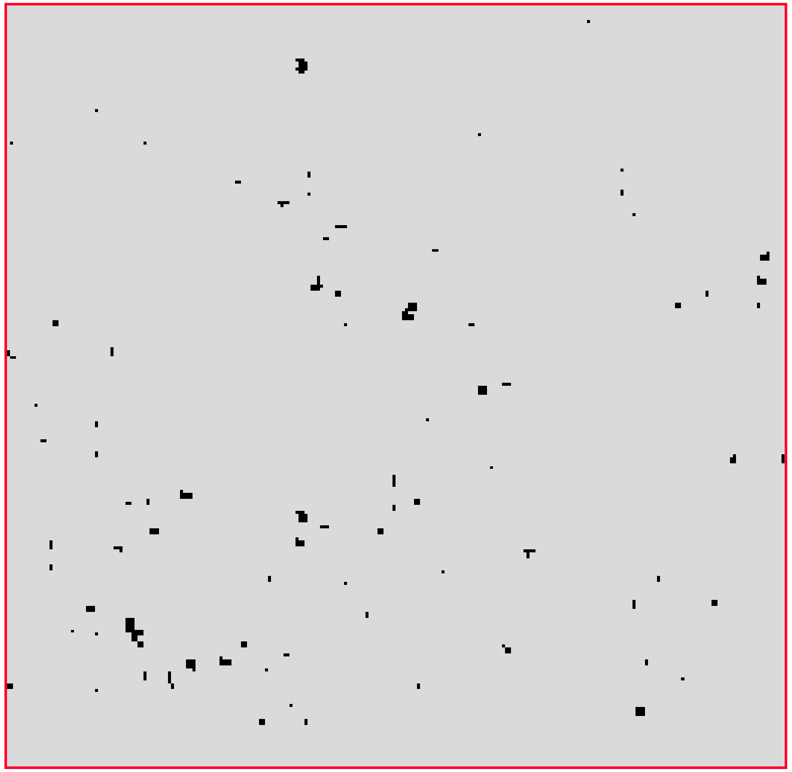

Partial map of the city of Nice

Houses with a Large Garden in Nice

Comment : *Detail of the previous map, where one wish to know the components of the connection by dilation, and, among them, those which are also arwise connected.*

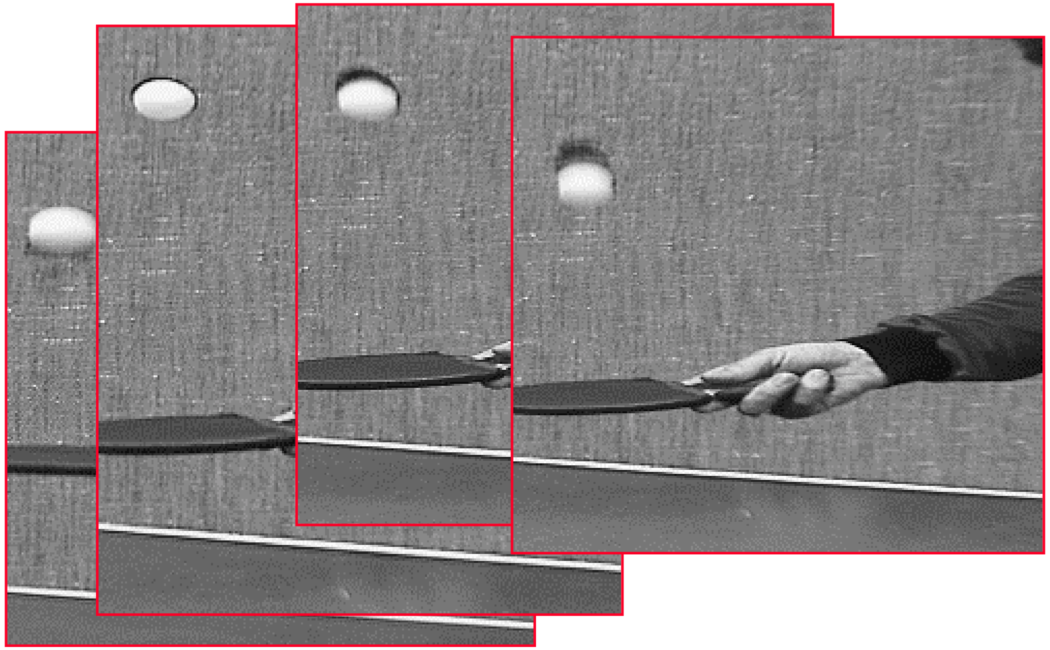


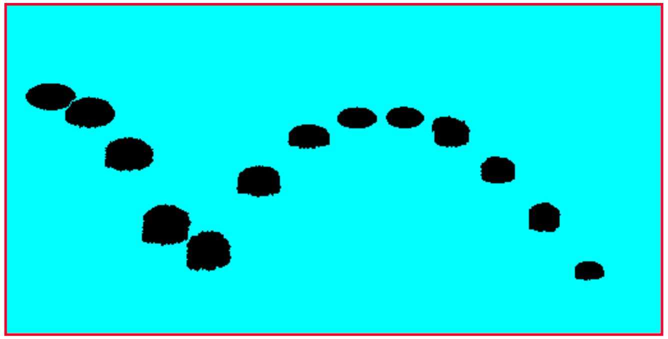
a) Components for the connection by dilation



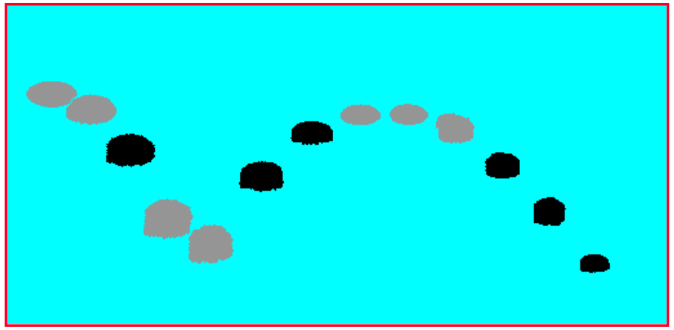
b : Isolated components of a) (according to the above algorithm)

Connections in a Sequence





b) representation of the ping-pong ball in the product Space⊗ Time



a) Extracts from an image sequence

c) Connections after a Space⊗ Time dilation of size 3 (in grey, the clusters)

Connected Operators

***Definition* :**

- Given a connection $\mathcal{C}$ on $\mathcal{P}(E)$, an operator $\psi : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ is said to be **connected** when it can only keep or suppress grains and pores of the set A under study.

  The most useful of such operations are those which, in addition, are **increasing**.

***Basic properties* :**

- All **binary** reconstruction increasing operations induce on the lattices of numerical functions, via the cross sections, increasing connected operators.
- Their possible properties to be strong filters, to constitute semi-groups, etc.. are transmitted to the connected operators induced on functions.

Connection and Reconstruction Opening

Connection allow to express, and to generalise reconstruction openings as follows

1) Call **increasing binary criterion** any mapping c: $\mathcal{P}(E) \to \{0,1\}$ such that:

$$A \subseteq B \;\Rightarrow\; c(A) \leq c(B)$$

2) With each criterion c associate **the trivial opening** γ_T: $\mathcal{P}(E) \to \mathcal{P}(E)$

$$\gamma_T(A) = A \quad \text{if} \quad c(A) = 1$$
$$\gamma_T(A) = \varnothing \quad \text{if} \quad c(A) = 0$$

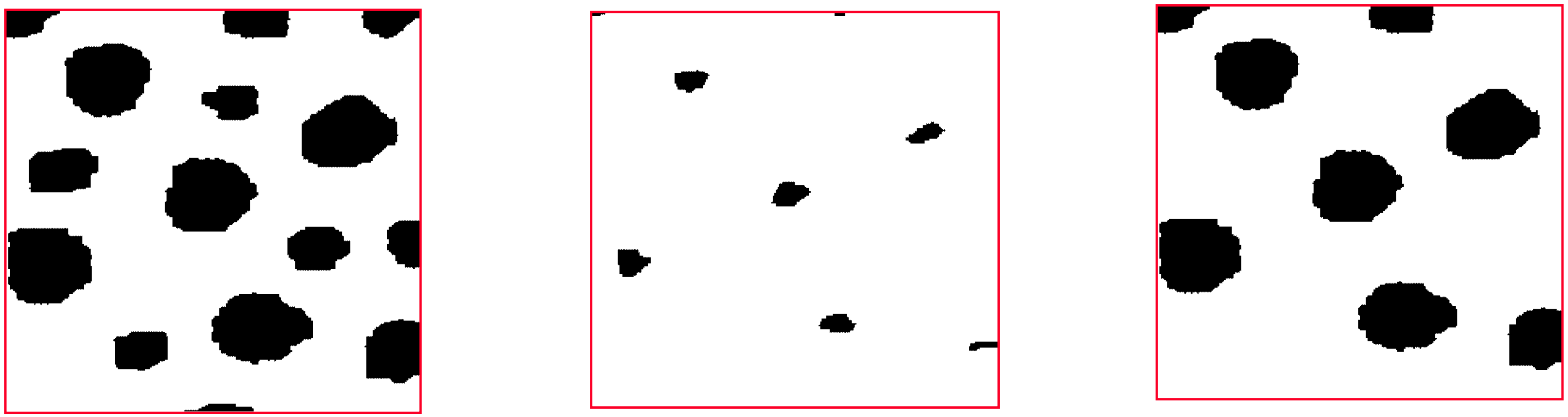
3) By generalising the geodesic case, we will say that γ^{rec} is **a reconstruction opening** according to criterion c when :

$$\gamma^{rec} = \vee\{\gamma_T \gamma_x,\, x \in E\}$$

γ^{rec} acts independently on the various components of the set under study, by keeping or removing them according as they satisfy the criterion, or not (*e.g. area, Ferret diameter, volume*..)

Application: Filtering by Erosion-Recontruction

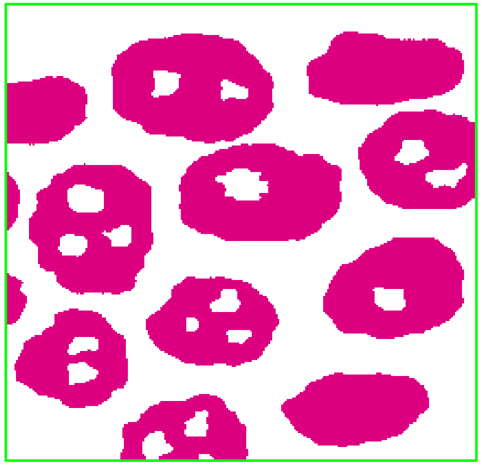
- Firstly, the erosion $X \ominus B_\lambda$ suppresses the connected components of X that cannot contain a disc of radius λ;
- then the opening $\gamma^{rec}(X ; Y)$ of marker $Y = X \ominus B_\lambda$ «re-builts» all the others.

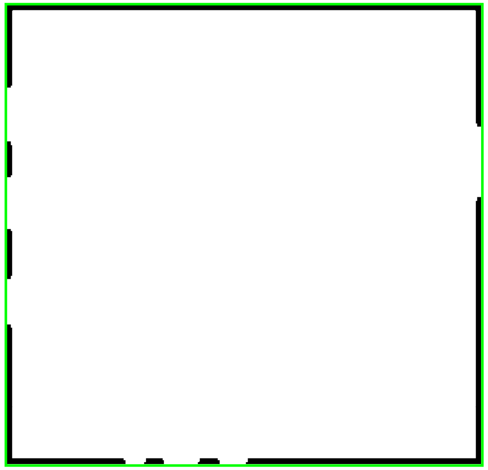


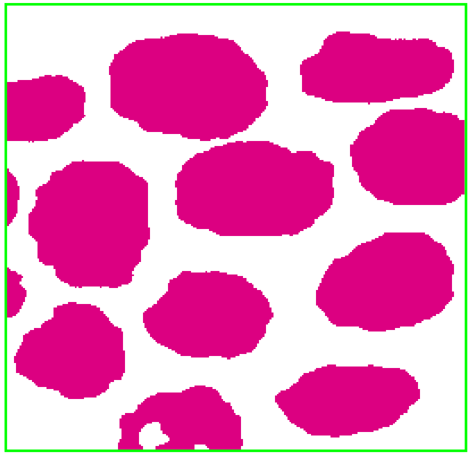
a) Initial image *b) Eroded of a) by a disc* *c) Reconstruction of b) inside a)*

Application: Holes Filling

Comment : *efficient algorithm, except for the particles that hit the edges of the field.*







initial image
X

A = part of the edge that hits X^C

reconstruction of A inside X^C

Closing by Reconstruction ; Lattices

- ***The closing by reconstruction*** $\varphi^{rec} = \complement\gamma^{rec}\complement$ is defined by duality.

  For example, in R^2, if we take the criterion

  - « *to have an area ≥10* », then $\varphi^{rec}(A)$ is the union of A and of the pores of A with an area ≤10;

  - or the criterion « *to hit a given marker M* », then $\varphi^{rec}(A)$ is the union of A and of the pores of A included in M^c .

- ***Associated Lattices:*** We now consider a family $\{\gamma^{rec}{}_i\}$ of openings by reconstruction, of criteria $\{ c_i \}$.Their inf $\cap\ \gamma^{rec}{}_i$ is still an opening by reconstruction, where each grain of A which is left must fulfil all criteria c_i , and where the sup $\cup\ \gamma^{rec}{}_i$ is the opening where one criterion at least must be satisfied (dual results for the closings). Hence we may state:

- ***Proposition:*** Openings and closing by reconstruction constitute two complete lattices for the usual sup and inf .

Strong Filters by Reconstruction

Here are a few properties of the filters by reconstruction

- ***Proposition****(J.Serra)* **:** Let γ^{rec} be a reconstruction opening on $T^{\mathbf{E}}$ that does not create pores and φ^{rec} be the dual of such an opening (not necessarily γ^{rec}). Then :

  $\boldsymbol{\nu = \varphi^{rec}\gamma^{rec}}$ **and** $\boldsymbol{\mu = \gamma^{rec}\varphi^{rec}}$ are **strong filters.**

  In particular, $I \wedge \gamma^{rec}\varphi^{rec}$ is an **opening**

- ***Proposition*** *(J.Crespo, J.Serra)* **:** Let $\{\gamma_i^{rec}\}$ and $\{\varphi_i^{rec}\}$ denote a granulometry and a (not necessarily dual) anti-granulometry, then

- the corresponding alternating sequential filters N_i and M_i are **strong** ; and

- both operators $\boldsymbol{\Psi_n = \wedge\{\varphi_i\gamma_i, 1 \leq i \leq n\}}$ and $\boldsymbol{\Theta_n = \vee\{\gamma_i\varphi_i, 1 \leq i \leq n\}}$ are **strong filters**.

Semi-groups of filters by Reconstruction

- ***Proposition** (Ph. Salembier, J.Serra)*: Let γ^{rec} be a reconstruction opening on E and φ be a closing that does not create particles. Then :

$$\varphi \gamma^{rec} \leq \gamma^{rec} \varphi \quad (\Leftrightarrow \gamma^{rec} \varphi \gamma^{rec} = \varphi \gamma^{rec} \Leftrightarrow \varphi \gamma^{rec} \varphi = \gamma^{rec} \varphi)$$

- ***Proposition** (Ph. Salembier, J.Serra)*: Let $\{\gamma_i^{rec}\}$ be a granulometry and $\{\varphi_i\}$ be an anti-granulometry of the above types. Then:

a) for all i, both products $\nu_i = \varphi_i \gamma_i^{rec}$ and $\mu_i = \gamma_i^{rec} \varphi_i$ satisfy the relations

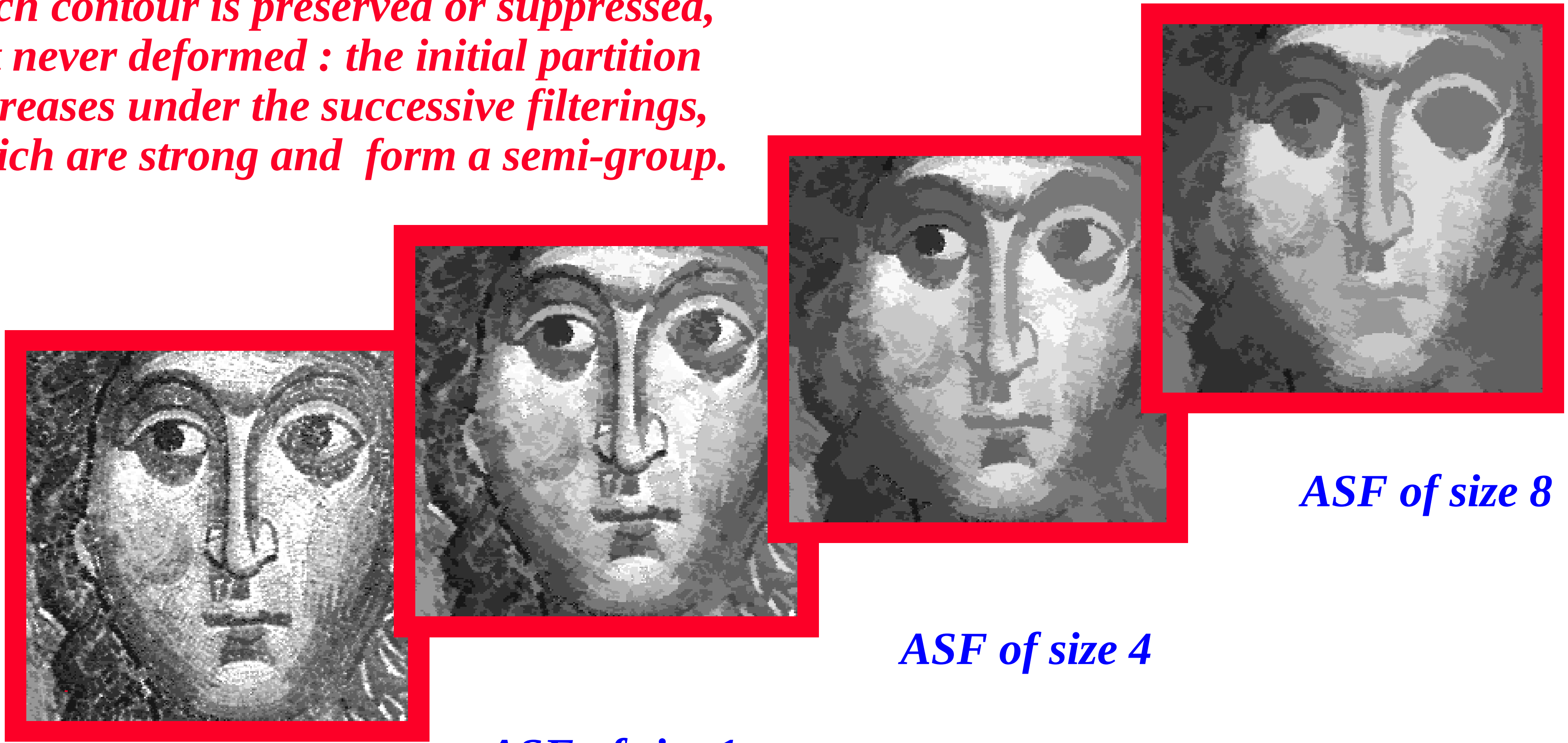
$$j \geq i \quad \Rightarrow \quad \nu_i \nu_j = \nu_j \quad \text{and} \quad \mu_i \mu_j = \mu_j$$

b) Therefore, the associated A.S.F. N_i et M_i form a **semi group**

$$N_j N_i = N_i N_j = N_{sup(i,j)} \quad ; \quad M_j M_i = M_i M_j = M_{sup(i,j)}$$

An Example of a Pyramid of Connected A.S.F.'s

Flat zones connectivity, (i.e. $\varphi = 0$).
Each contour is preserved or suppressed,
but never deformed : the initial partition
increases under the successive filterings,
which are strong and form a semi-group.



ASF of size 8

ASF of size 4

ASF of size 1

Initial Image

Levelling I

- Markers based openings allow to design a *self-dual* operator, called levelling, and due to *F.Meyer*. Let

~ $\gamma_M(A)$ be the union of the grains of A that hit M ou that are adjacent to it (*i.e.* disjoint from M but whose union with a grain of M is connected)

~ $\varphi_M(A)$ be the union of A and of its pores that are included in M and non adjacent to M^c

- Then take the ***activity supremum***

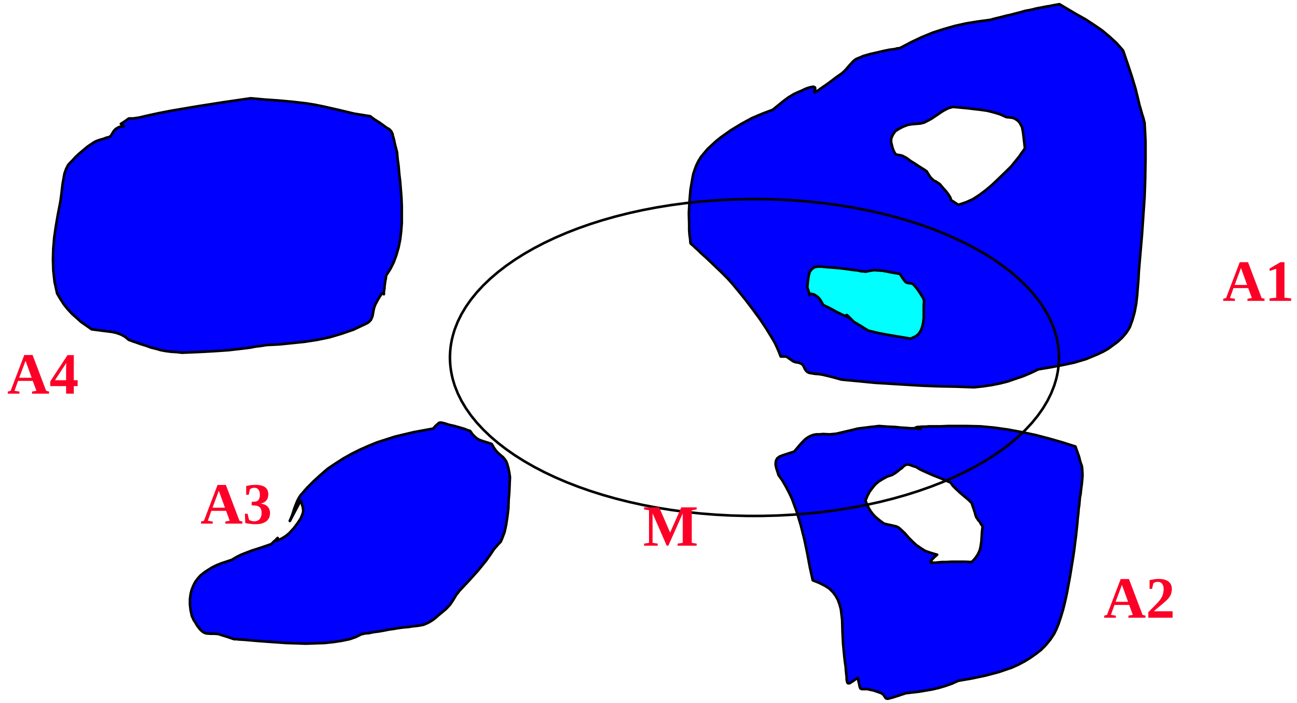
$$\lambda = \gamma_M \curlyvee \varphi_M$$

i.e. $\lambda(A) \cap A = \gamma_M \cap A$, and $\lambda(A) \cap A^c = \varphi_M \cap A^c$.

Levelling λ acts inside A as the opening, and inside A^c as the closing.

- ***Self-duality:*** The mapping $(A,M) \rightarrow \lambda(A,M)$ from $\mathcal{P}(E) \times \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ is self-dual. If M itself depends on A, *i.e.* if $M = \mu(A)$, then the levelling, as a function of A only, is self-dual if and only if μ is already self-dual.

Levellings II



- The levelling of marker M extracts: grain A1, with one of its pores ;
  grain A2, without its pore ;
  grain A3 .

Levellings III

Here are a few nice properties of levelling :

- ***Proposition (F.Meyer):*** The levelling (A,M)→ λ(A,M) is an increasing mapping from $\mathcal{P}(E)\times \mathcal{P}(E)\rightarrow\mathcal{P}(E)$; it admits the equivalent expression:

$$\lambda = \gamma_M \cup (\complement \cap \varphi_M)$$

- ***Proposition (G.Matheron):*** The two mappings

  ~ $A\rightarrow \lambda_M(A)$, given M, and

  ~ $M\rightarrow \lambda_A(M)$, given A, are idempotent.

- ***Proposition (J.Serra):*** The levelling $A\rightarrow \lambda_M(A)$ is a strong filter, and is equal to the commutative product of its two primitives

$$\lambda = \gamma_M \circ \varphi_M = \varphi_M \circ \gamma_M$$

  Therefore, it satisfies the stability relation : $\gamma_x(I \cup \lambda) = \gamma_x \cup \gamma_x \lambda$, which preserves the *sense of variation* at the grains/pores junctions

Example of Levelling, I

Initial image : *« Joueur de fifre », by E. MANET*

Markers : *Square alternated sequential filters, size 2 (non self-dual)*

Initial image, 83.776 pp
flat zones : 34.835

Marker $\varphi\,\gamma$
flat zones : 53.813

Marker $\gamma\,\varphi$
flat zones : 53.858

Example of Levelling, II

Marker: ***extrema with a dynamics ≥ h (marker invariant under duality).***

Initial image
flat zones : 34.835

h = 80
flat zones : 57.445

h = 110
flat zones : 65.721

Example of Levelling, III

Marker: ***Initial image, where the h-extrema are given value zero (self-dual marker)***

Initial image
flat zones : 34.835

h = 50
flat zones : 58.158

h = 80
flat zones : 59.178

Example of Levelling, IV

Marker: ***Gaussian convolution of size 5 of the noisy image***

A :initial image, with
10.000 noise points

B : Gaussian
convolution of A

A levelled by B
flat zones : 46.900

Connectivity and Reconstruction

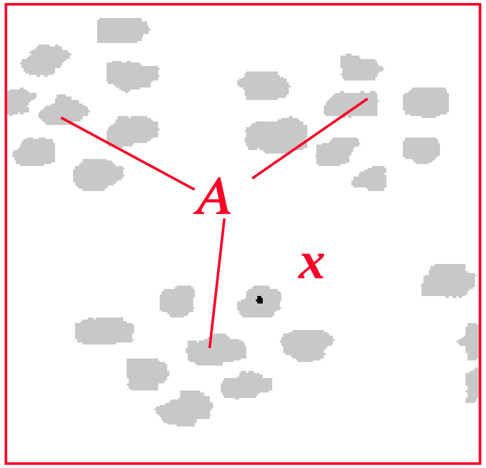
- We saw that if point x is a marker and A a set, the infinite geodesic dilation $\cup\, \delta_A^{(n)}(x)$ leads to the point connected opening of A at x

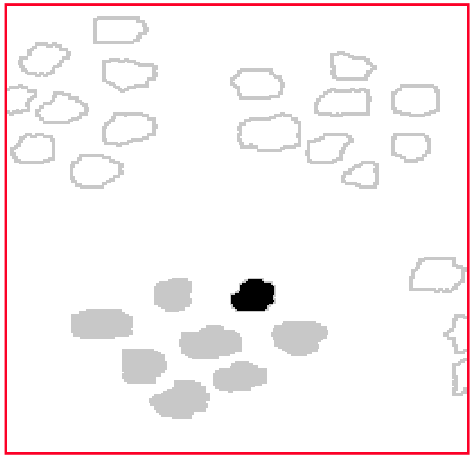
$$\gamma_x(A) = \cup\, \delta_A^{(n)}(x) \qquad (1)$$

- What happens when we replace the unit disc δ by that of radius 10, for example, in Eq. (1) ? Obviously, **clusters** of particles are created. Here two questions arise:

  1- Do we obtain a ***new connection,*** *i.e.* which still ***segments*** set A ?

  2- Must we operate by means of ***dilations*** according to ***discs*** ?





Geodesy et Connections

Curiously, the answer to these questions depends on properties of symmetry of the operators. A mapping $\psi : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ is ***symmetrical*** when

$$\mathbf{x} \subseteq \psi(\mathbf{y}) \quad \Leftrightarrow \quad \mathbf{y} \subseteq \psi(\mathbf{x})$$

for all points x,y de E.

- ***Proposition*** *(J.Serra)* **:** Let $\delta: \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ be an extensive and symmetrical dilation, and let $x \in E$, et $A \in \mathcal{P}(E)$. Then the limit iteration

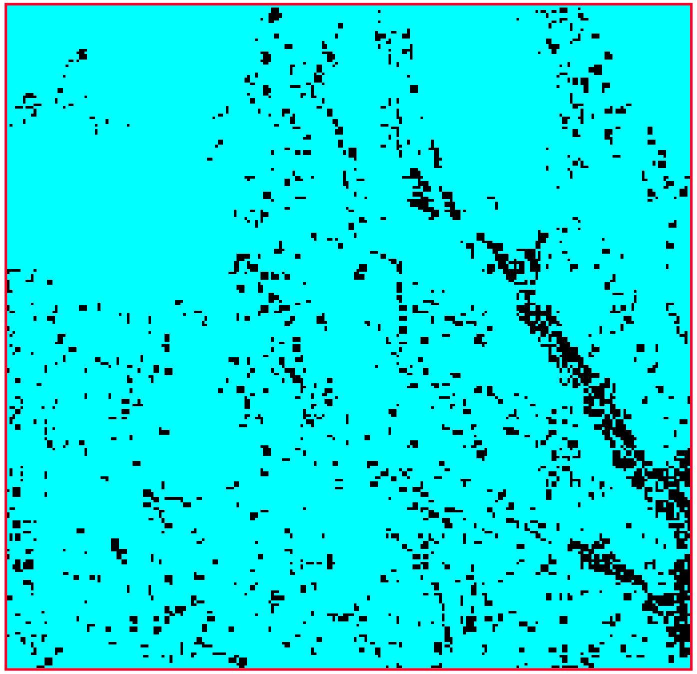
$$\gamma_x(A) = \cup \{\delta_A^{(n)}(x), n > 0\}$$

considered as an operation on A, is a **point connected opening**.

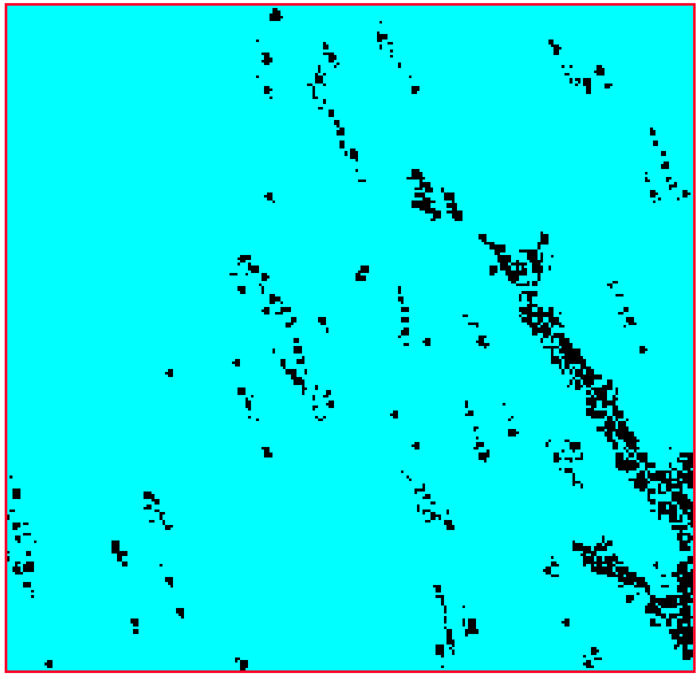
Note that the starting dilation δ does not need itself to be connected !

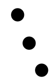
Nice : Directional Alignments

Comment ***: Although the structuring element D used for the reconstruction is not connected, it generates a new connection. For display reasons, we the smaller components have been filtered out .***



a) Zone A under study



b) Reconstruction of A from $A \ominus 2B$
by means structuring element D = 
where each point indicates a unit hexagon

References

***On Binary Connections* :**

- Morphological connectivity for sets was introduced by J.Serra and G.Matheron for designing strong filters {SER88,ch7}. The characteristic connected opening and the connections of second generation appear also for the first time in {SER88, ch2}. In {RON98}, Ch.Ronse proposes equivalent axiomatic, which emphasises another point of view, and he provides number of instructive examples.

***On Connected Operators* :**

- In {MEY90} and in {SAL92}, reconstruction is used as a tool to modify the homotopy of a function, for multi-resolution purposes. The contrast opening is defined in {GRI92}. A systematic investigation of semi-groups and pyramids, by Ph.Salembier and J.Serra, is given in {SER93a} and used for sequences compression and filtering in {MGT96}, {SAL96}, {PAR94}, {CAS97}, and {DEC97}. Nice properties of $\vee$ and $\wedge$ were found by J.Crespo and Al {CRE95}.

- The theory of leveling is due to F.Meyer {MEY98}, G.Matheron {MAT97}, and J.Serra {SER98b}. The larger class of the "grains operators" has been introduced and studied by H. Heijmans {HEI97}.

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