Haudorff distances and Interpolations

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Hausdorff Distance

- E is a metric space of distance d, and $\mathcal{K}'$ is the class of the non empty compact sets of E. Put :

$$\mathbf{d\,(x, Y) = \inf \{ d(x,y),\; y \in Y \}\, ; \qquad x \in E \quad Y \in \mathcal{K}'}$$

Then the mapping $\mathcal{K}' \times \mathcal{K}' \rightarrow R_+$

$$\boldsymbol{\rho}\mathbf{(X,Y) = \max \left\{ \sup d\,(x, Y)\, ;\, \sup d\,(x, Y) \right\}} \qquad \textit{(Eq. 1)}$$

is a distance, called **«Hausdorff Distance»**, on $\mathcal{K}'$.

- By introducing the dilation δ_λ by the compact ball B_λ (x) of centre x and radius λ , *i.e.*

$$\boldsymbol{\delta_\lambda\,(X) = \cup\{B_\lambda\,(x),\, x \in X\}}$$

(Eq. 1) takes the following form

$$\boldsymbol{\rho(X,Y) = \inf \left\{ \lambda : X \subseteq \delta_\lambda\,(Y)\, ;\, Y \subseteq \delta_\lambda\,(X) \right\}} .$$

First Haudorff Geodesic

- If it exists, a geodesic between X and Y will be a shortest segment [X,Y] in space $\mathcal{K}'(E)$, *i.e.* a family { Z_α , $0 \leq \alpha \leq 1$ } of non empty compact ***interpolators*** from X , for $\alpha = 0$, to Y, for $\alpha = 1$.

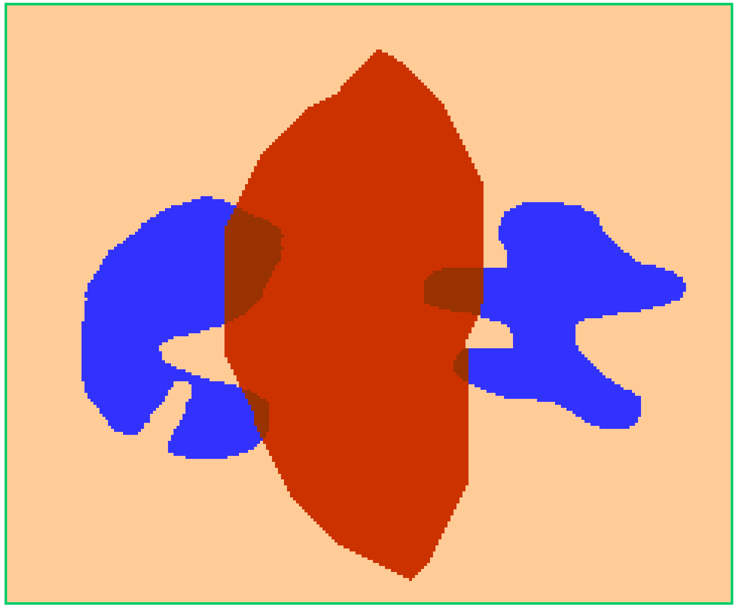
- ***Proposition (1rst geodesic in $\mathcal{K}'$) :*** Every pair (X,Y) in $\mathcal{K}'(E)$, from haudorff distance ρ apart, admits the following geodesic:

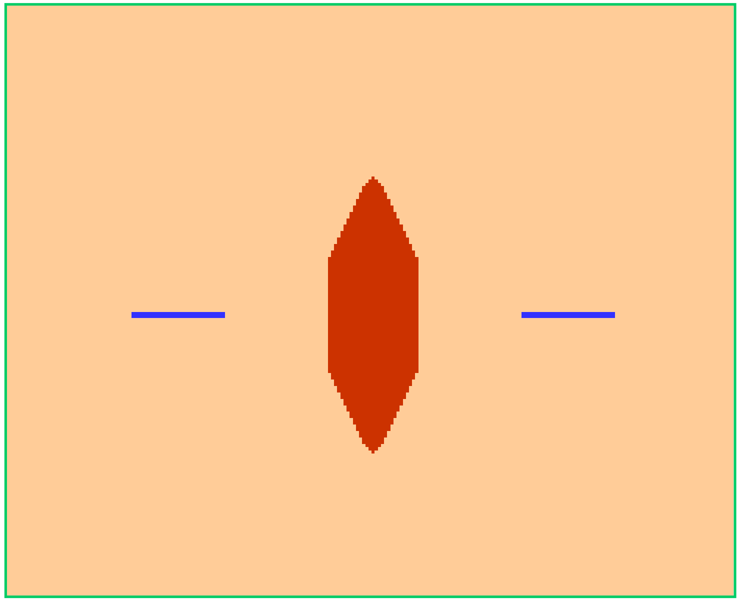
$$\{ \; Z_\alpha = \delta_{\alpha\rho}(X) \cap \delta_{(1-\alpha)\rho}(Y) \; ; \; 0 \leq \alpha \leq 1 \; \}$$

- Set Z_α turns out to be the intersection of the dilates of X and of Y by the balls of radii $\alpha\rho$ and $(1-\alpha)\rho$ respectively.

  In particular, in Minkowki case, $X \oplus B(\rho/2) \cap Y \oplus B(\rho/2)$ is the ***midway set*** between X and Y.

Two Examples of Midway Sets



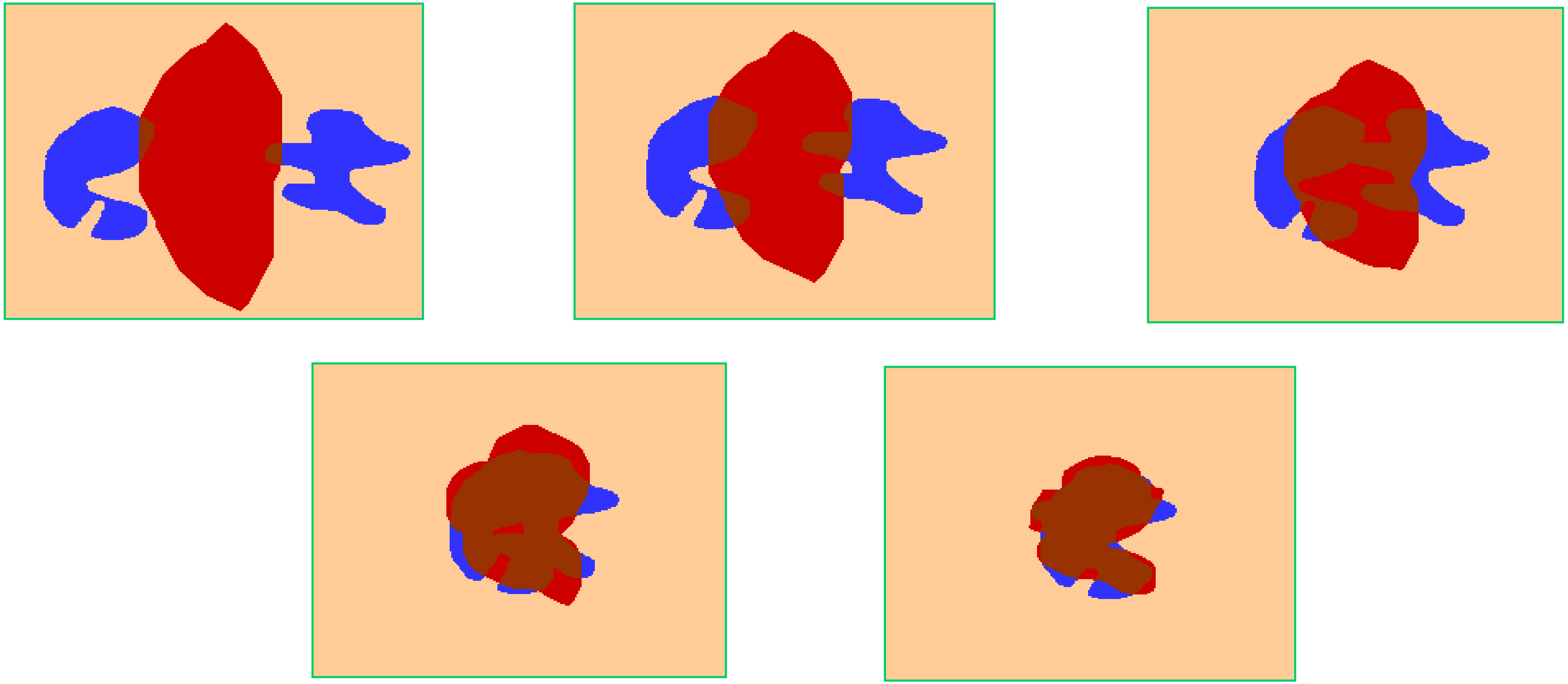


Comments : In both examples, the geodesic has a ***swelling*** effect. In the second one, two ***fine*** and ***horizontal*** segments are interpolated by a ***thick vertical*** lens !

Questions : 1/ Should it be possible to approach separately the relative ***positions*** of X and Y, and their ***shape*** differences ?

2/ Is the above geodesic the unique one ?

Translation Effect on $Z_{0.5}$



As the two sets diverge, their geodesic Z_α becomes less and less significant .

Reduced Hausdorff Distance

- ***Reduced space :*** Let E be a compact sub-space of R^n or Z^n . We will approach locations and shapes separately, by considering the quotient space $\mathcal{K}_1$ of $\mathcal{K}'$ for the equivalence under translation (Notation : $\mathbf{X_a}$ stands for the translate of X by vector a) . Put

$$\rho_1 (X,Y) = \inf \{ \rho (X_u , Y_v) , \quad u,v \in E \} \qquad \textit{Eq(2)}.$$

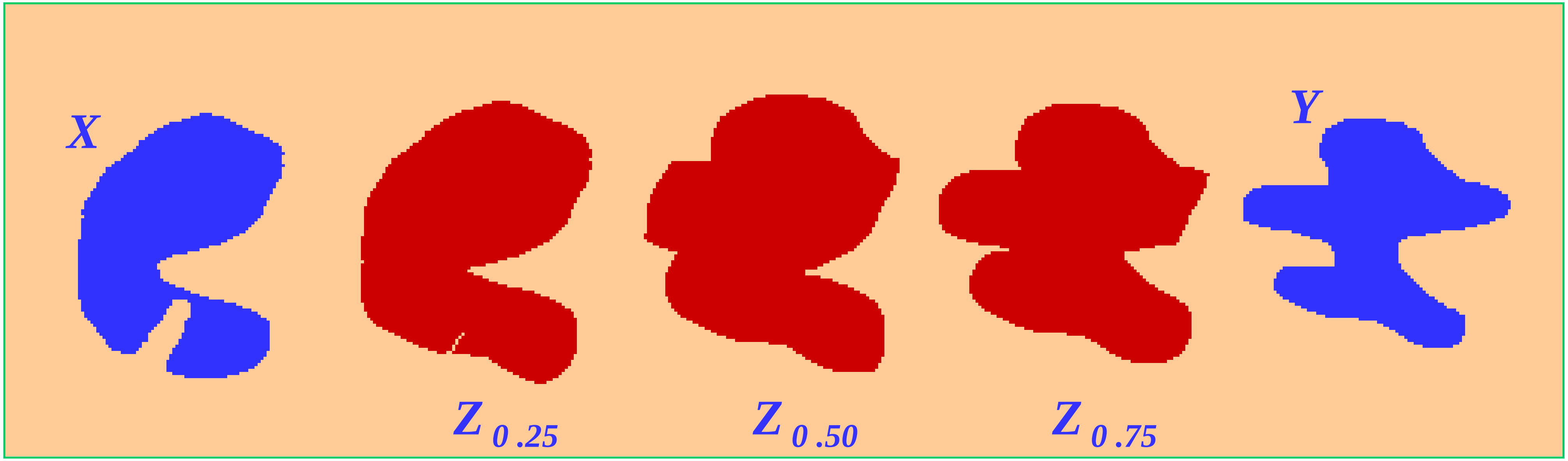
Since space E is compact, there exists at least one pair (X_a , Y_b) for which $\rho = \rho_1$, and this yields the following result

- ***Proposition (1rst geodesic on $\mathcal{K}_1$) :*** The mapping introduced by ***Eq.(2)*** defines a ***distance*** on the quotient space $\mathcal{K}_1$. Moreover, for every pair of compact sets X,Y , the geodesic in $\mathcal{K}_1$ is nothing but the (non reduced) geodesic of X_a , Y_b in $\mathcal{K}'$ *i.e.*

$$\{ \; Z_\alpha = X_a \oplus B\alpha\rho \;\cap\; Y_b \oplus B(1-\alpha)\rho \;;\; 0 \leq \alpha \leq 1 \; \}$$

In practice, a matching of the centres of X and Y is sufficent.

Reduced Distance : an Example



- The geodesics were computed when the centers of gravity of X and of Y were superimposed (on the figure, set Y is shifted for display reasons).
- The three intermediary Z_α correspond to $\alpha = \{ 0.25 ; 0.50 ; 0.75 \}$
- The residual swelling effect is more acceptable.

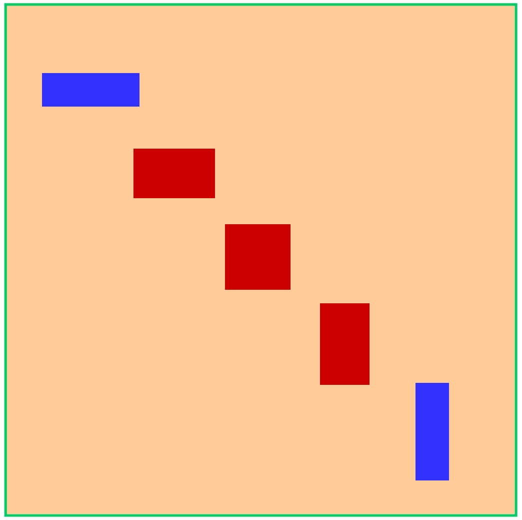
Haudorff Geodesic for Convex Sets (I)

A second way to improve the geodesics is suggested by the convex sets.

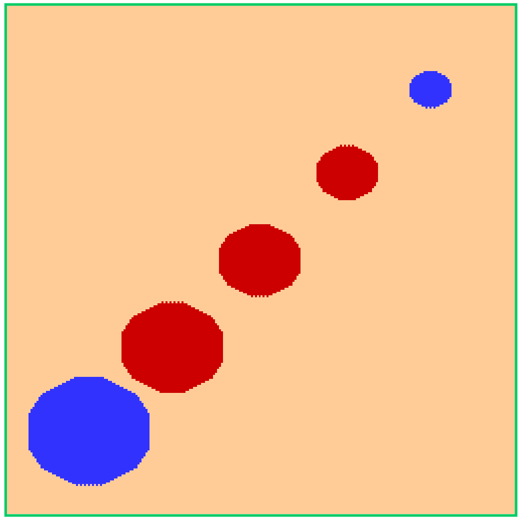
- ***Convex case:*** Take for E the Euclidean space R^n , and focus on the metric sub-space $\mathcal{C}' \subseteq \mathcal{K}'$ of the ***convex*** compact sets. then we have :

- ***Proposition (Geodesics on $\mathcal{C}'$):*** let X and Y be two ***convex*** compact sets in R^n , then the interpolators $\{C_\alpha\}$ form a ***geodesic*** in space $\mathcal{C}'$.

$$\{C_\alpha\} = \{(1 - \alpha)X \oplus \alpha Y ,\ 0 \leq \alpha \leq 1\}$$



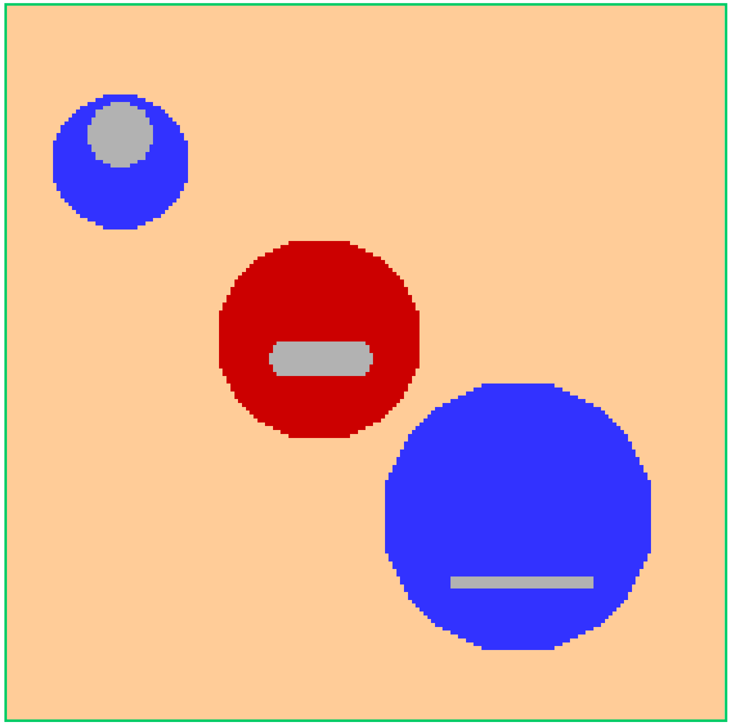
Examples of geodesics C_α



Haudorff Geodesic for Convex Sets (II)

Properties of geosdesic C_α

- Unlike the first geodesic Z_α , C_α ***commutes under translation,*** *i.e.* when X is shifted by a, then C_α (X,Y) is shifted by α.a ;
- Over *C* ', geodesic **C_α** is always smaller than **Z_α** *i.e.* **$C_\alpha \subseteq Z_\alpha$;**
- The mapping $C_\alpha : R^n \times R^n \rightarrow R^n$ is ***increasing*** ;
- But when X and Y are ***not*** in *C* ' then C_α is no longer a geodesic !



Increasingness of Geodesic C_α

Second Haudorff Geodesic : General Case

- ***Proposition (Second Geodesic on $\mathcal{K}$ ') :*** Every pair (X,Y) in $\mathcal{K}$ '(E), from haudorff distance ρ apart, admits the following geodesic:

$$\{ Z'_\alpha = \delta_{\alpha\rho}(X) \cap \delta_{(1-\alpha)\rho}(Y) \cap (1-\alpha)X \oplus \alpha Y\ ; \quad 0 \le \alpha \le 1 \ \} ;$$

Hence, by comparison with the first geodesic $Z_\alpha = \delta_{\alpha\rho}(X) \cap \delta_{(1-\alpha)\rho}(Y)$, we now have:

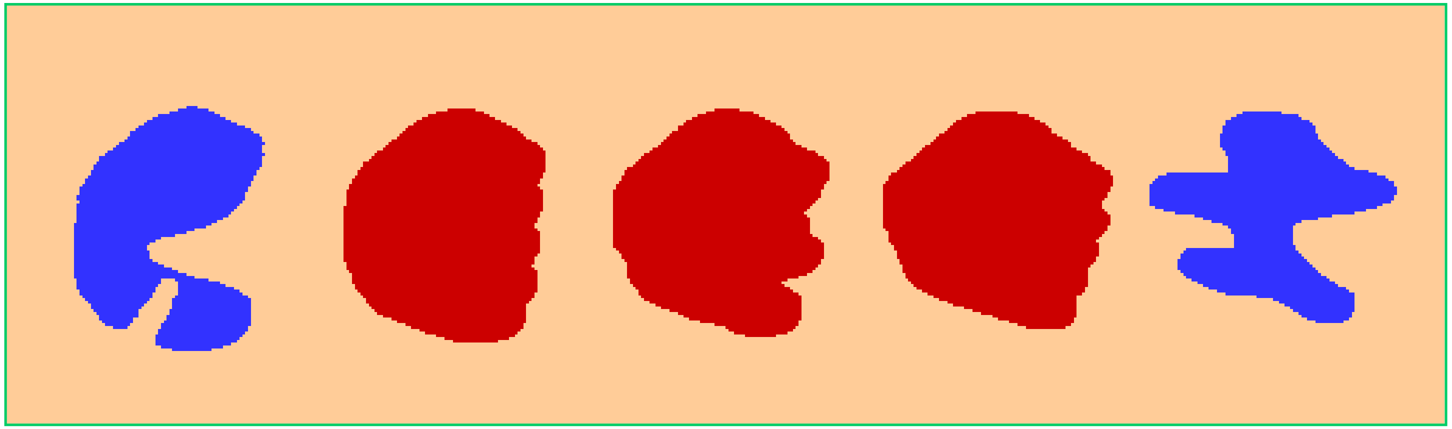
$$Z'_\alpha = Z_\alpha \cap C_\alpha$$

- ***Comment : 1/*** Here, not only X and Y are not necessarily convex, but space E itself is no longer supposed to be Euclidean.

  2/ Since C_α commutes under translation, the above reduced approach is still valid : given the pair (X,Y) and their optimal translates (X_a, Y_b), family $\{Z'_\alpha(X_a, Y_b);\ 0 \le \alpha \le 1\}$ is a ***geodesic*** on the reduced space $\mathcal{K}_1$.

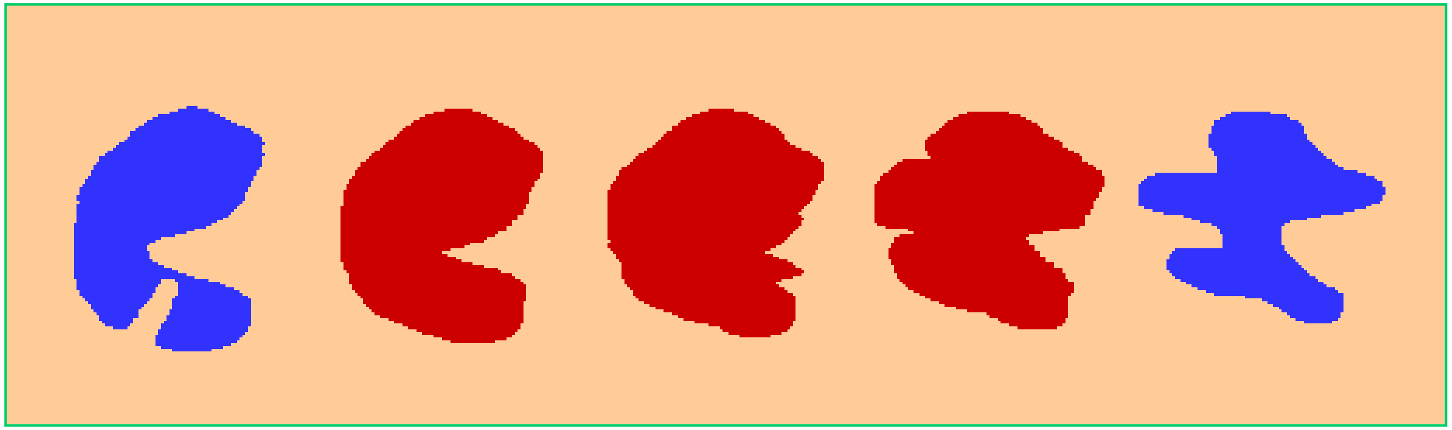
Comparison between C_α and $Z_\alpha \cap C_\alpha$

(I) shapes and sizes



series C_α

Pseudo- geodesic C_α : the shape evolution is not well caught.

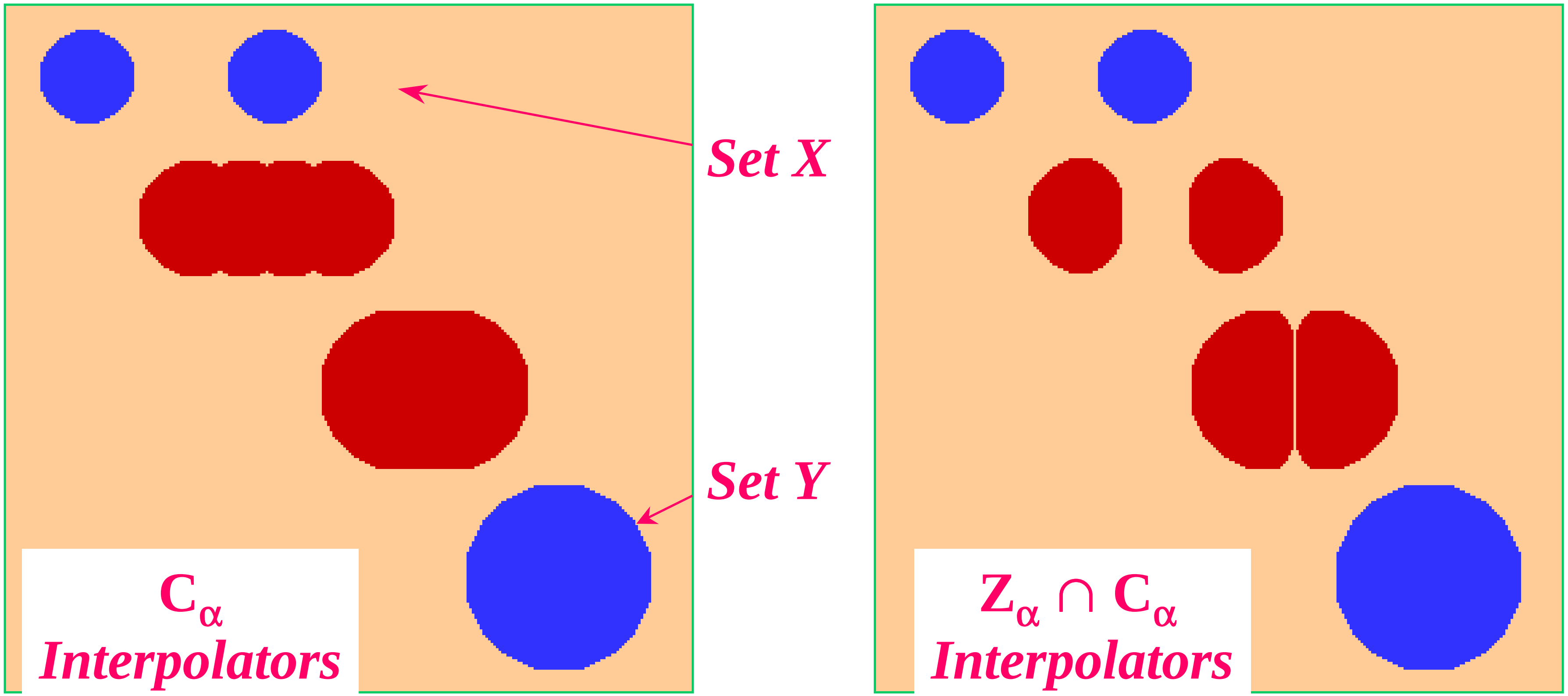


series

$Z'_\alpha = Z_\alpha \cap C_\alpha$

Reduced 2nd geodesic Z'_α : both swelling effect and shape evolution are improved.

Comparison between C_α and $Z_\alpha \cap C_\alpha$



(II) Connectivity

When one input at least is not convex, then C_α is no longer a geodesic (*e.g.* $C_\alpha(X, X)$ is not X) and yields less satisfactory results than $Z_\alpha \cap C_\alpha$.

Comparison between C_α and $Z_\alpha \cap C_\alpha$

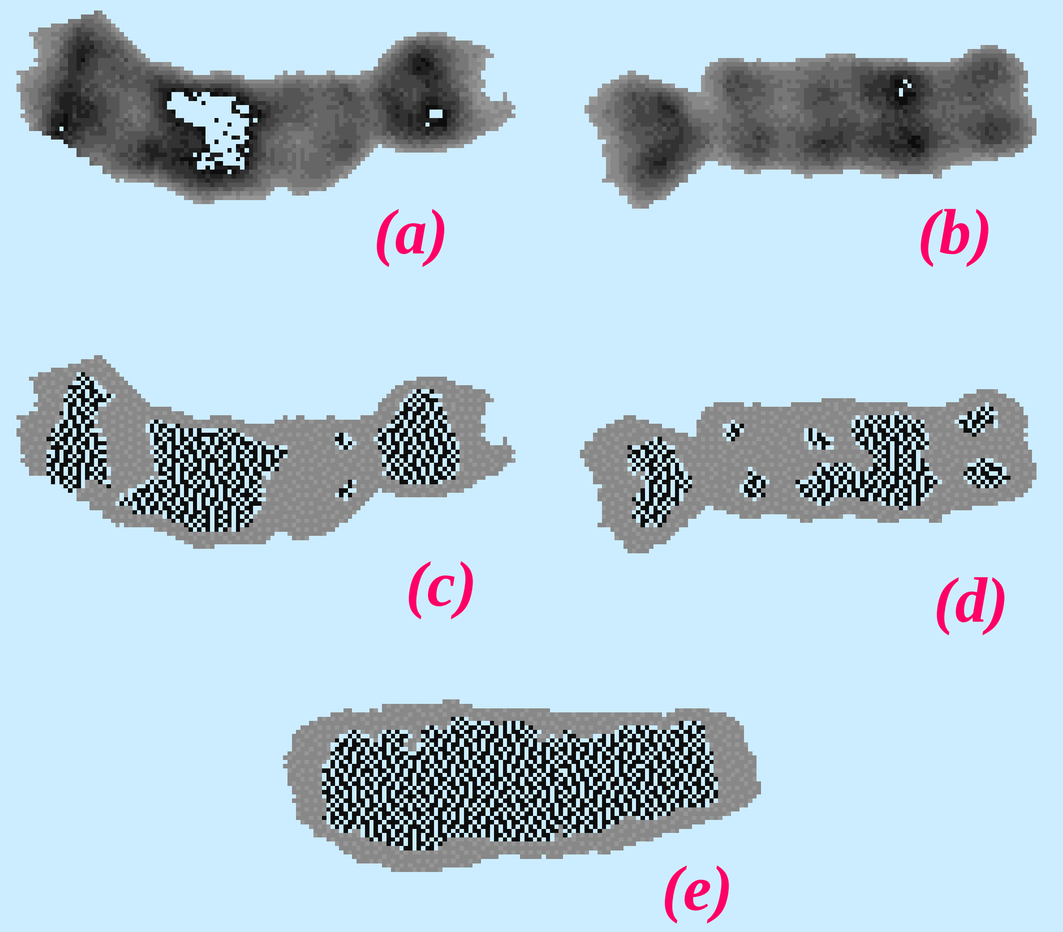
(III) Connectivity

However, the nice previous connectivity preservation fails when as soon as homotopy becomes more complicated.

(a) (b) two chromosoms ;

(c) (d) basic threshold of the bending ;

(e) Midway set according to the 2nd geodesic $Z_\alpha \cap C_\alpha$

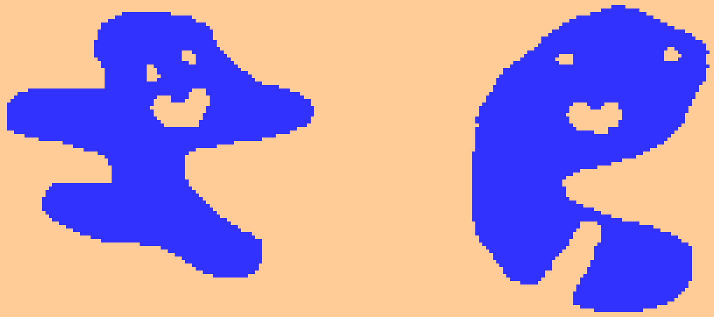


Comparison between C_α and $Z_\alpha \cap C_\alpha$

(IV) Increasingness

Unlike C_α , geodesic $Z_\alpha \cap C_\alpha$ is ***not increasing***.

Practically, what happens if we interpolate the homolog pairs individually (eyes and mouth) ?



Young ghosts smiling,

Comparison between C_α and $Z_\alpha \cap C_\alpha$

(IV) Increasingness

When the involved shapes are not too tortuous, then increasingness is preserved.

Here, eyes and mouth have been interpolated by using geodesic $Z_\alpha \cap C_\alpha$.

Hausdorff Distance by Erosions

Basically, the swelling effect arises because Hausdorff distance is not a ***self-dual*** notion. A first step to offset this weakness consists the following :

- ***Dual Hausdorff Metric :*** Consider the subclass of $\mathcal{K}'$ made of regular compact sets *i.e.* whose elements A satisfy the equality

$$\overline{\mathring{A}} = A$$

then the non negative number

$$\sigma(X,Y) = \inf \{ \lambda : \varepsilon_\lambda (X) \subseteq Y \; ; \; \varepsilon_\lambda (Y) \subseteq X \}$$

defines a ***Hausdorff Distance by Erosions*** on the regular class.

- ***Euclidean case :*** Below, we will focus on the class $\mathcal{A}$ of sets which are
  - regular in a compact subspace E of R^n or Z^n ;
  - finite unions of disjoint connected sets.

Interpolations for Nested Sets

Consider an ordered pair (X,Y) of sets in $\mathcal{A}$(E) , *e.g.* with $X \subseteq Y$.

- ***Median element :*** A point m lies at a distance $\leq\lambda$ from X iff $m \in (X \oplus \lambda B)$; similarly, by regularity of Y, m lies at a distance $\geq\lambda$ from Y^c iff $m \in (Y \ominus \lambda B)$; hence set

$$\mathbf{M(X,Y) = \bigcup \{ (X \oplus \lambda B) \cap (Y \ominus \lambda B) , \lambda \geq 0 \}} \qquad \textit{(Eq. 3)}$$

characterizes a median element such that

1/ $\mathbf{X \subseteq M \subseteq Y}$;

2/ $\partial\mathbf{M}$ is the locus of the points equidistant from X and from Y^c (the SKIZ of $X \cup Y^c$, in Lantuejoul's sense) ;

3/ all the involved distances are smaller or equal to

$$\mu = \mathbf{inf} \{ \lambda : \lambda \geq 0 , (X \oplus \lambda B) \cap (Y \ominus \lambda B)^c \neq \varnothing \} . \qquad \textit{(Eq. 4)}$$

Median Element and Haudorff Distances

- ***Compacity :*** Because of the assumptions of regularity and of finitude, the median element M (X,Y) belongs to $\mathcal{A}$(E) **,** and there exists at least one point z on **∂M** such that $B_\mu(z)$ hits both X and the closure of Y^c .

- ***Proposition (Median element and distances) :*** Given X,Y in $\mathcal{A}$(E) , the median element M (X,Y) is at Haudorff ***dilation*** distance from X and $X \bullet \mu B$ and also at Hausdorff ***erosion*** distance from Y and $Y \circ \mu B$.

Note that in these results, none of the distances between X and Y intervenes

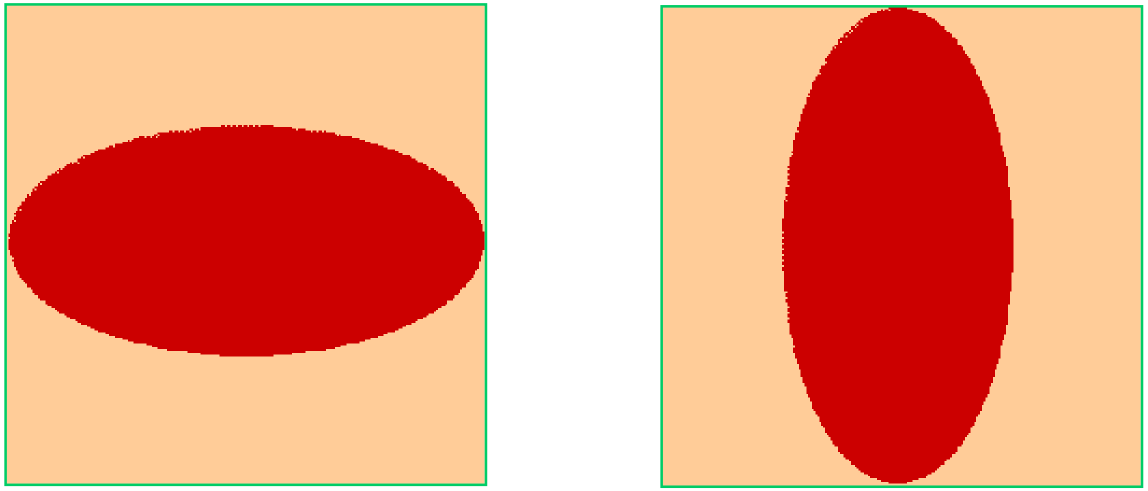
- ***Weighted element :*** By intoducing two weights α and (1 - α) in Eq. 2 we generalize M (X,Y) as follows :

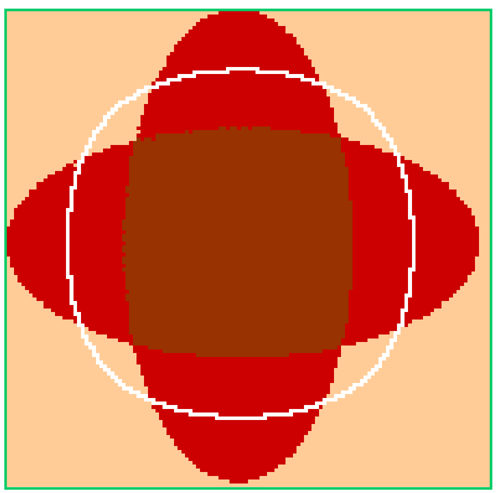
$$\mathbf{M_\alpha(X,Y) = \cup \{ (X\oplus \alpha\lambda B) \cap (Y\ominus (1 - \alpha)\lambda B) , \lambda \geq 0 \} \qquad 0\leq\alpha\leq 1}$$

to which is associated the minimum value μ(α), with $\sup_\alpha \{ \mu(\alpha) \} = \rho(X,Y)$.

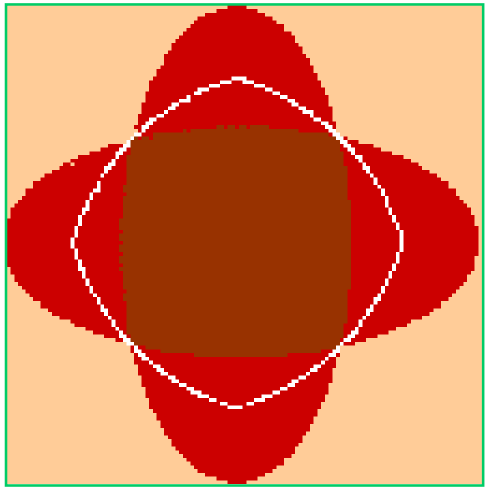
Examples of Median Elements

Initial sets

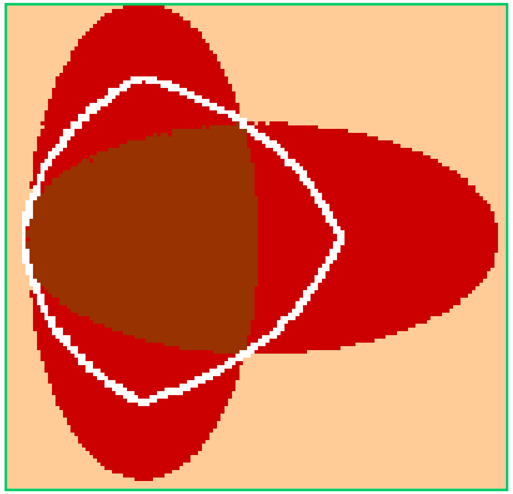




Midway set $C_{0.5}$ (commutes under translation)

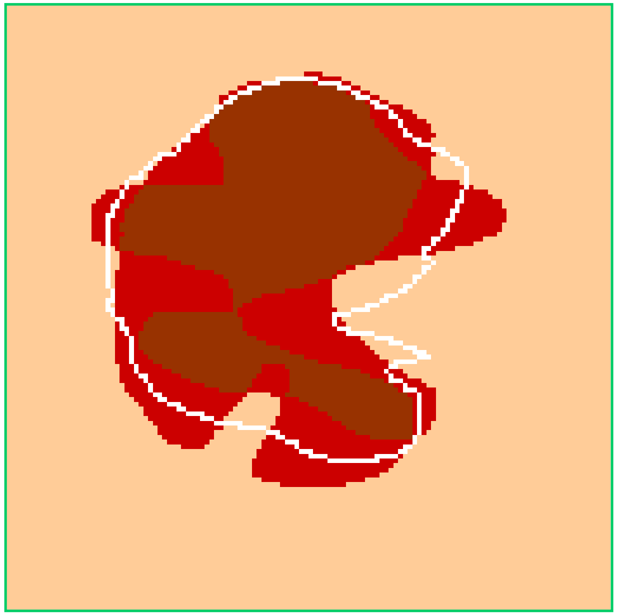
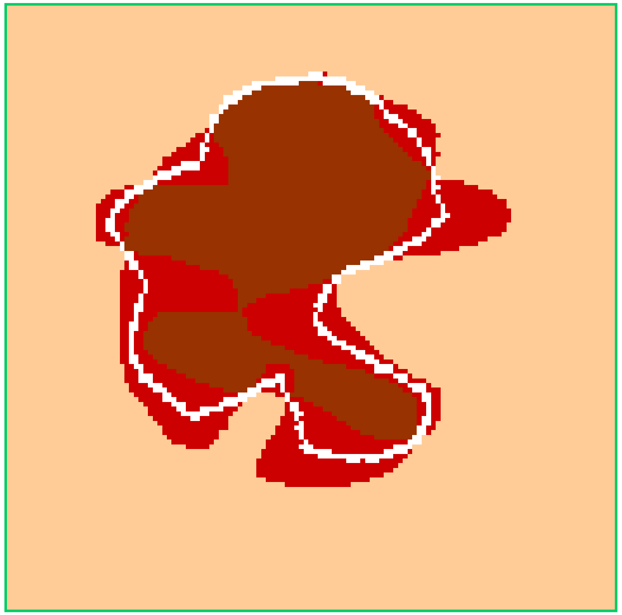


Middle element $M_{0.5}$



Middle element $M_{0.5}$ after shift of one set

Another Example

Conlusions :

1/ the M $_{\alpha}$'s are not geodesic sets : the midway between X and $M_{0.5}$(X,Y) is ***not*** $M_{0.25}$(X,Y) ;

2/ the translation dependence is worse for the M $_{\alpha}$'s than for the Z' $_{\alpha}$'s ;

3/ but (X,Y) →M $_{\alpha}$ (X,Y) is ***increasing***, hence it extends easily to ***numerical functions*** (see F. Meyer, S. Beucher and J.R. Casas works on the subject) .

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