Morphological Operators on the Unit Circle

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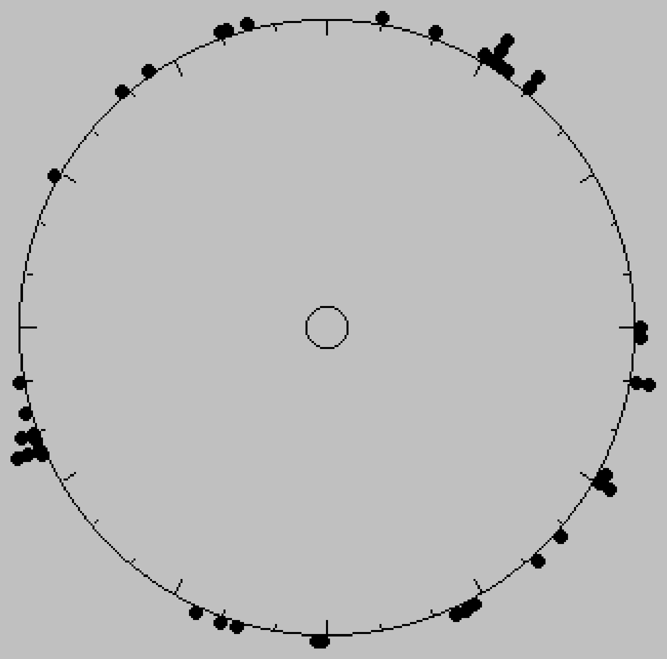
The unit circle

- In image analysis, one often has to treat data distributed on the unit circle
- Two examples are:
  - The hue band of colour images
  - Images describing directional texture



Hue band

Colour image – “The Virgin”, P. Serra (16th century)

Hue band

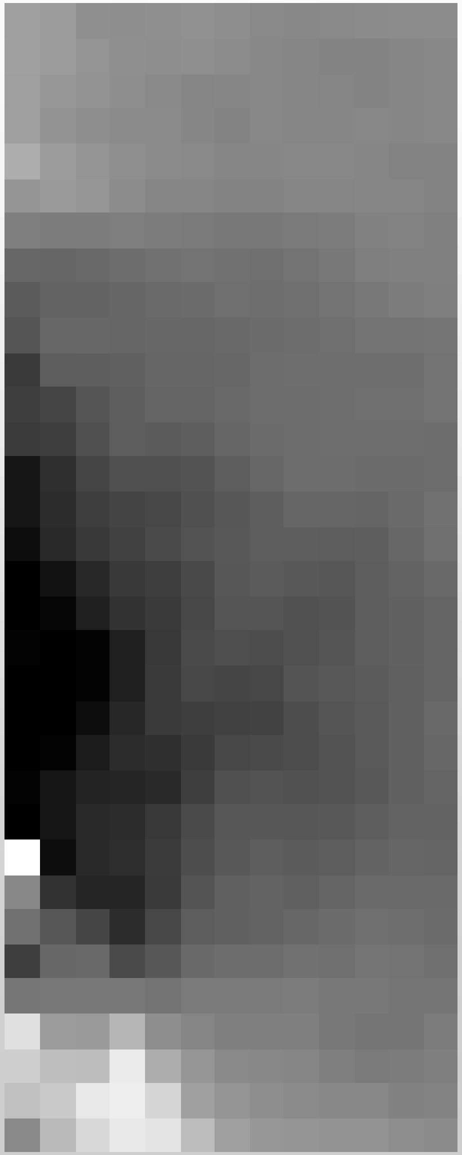
The red and violet colours are separated by a large discontinuity, even though they are visually “similar”

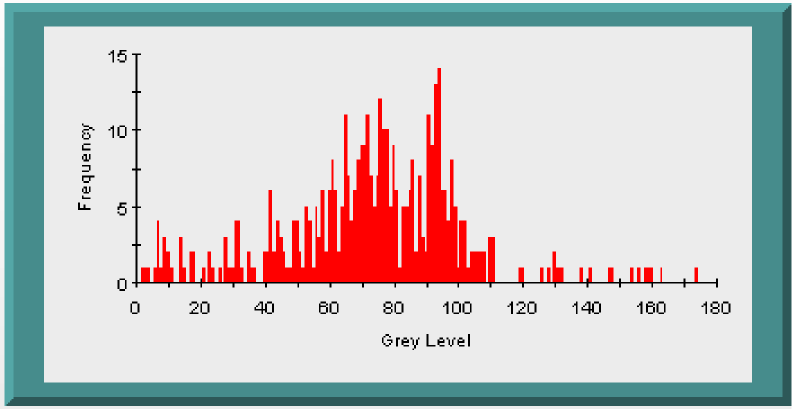
Hue histogram

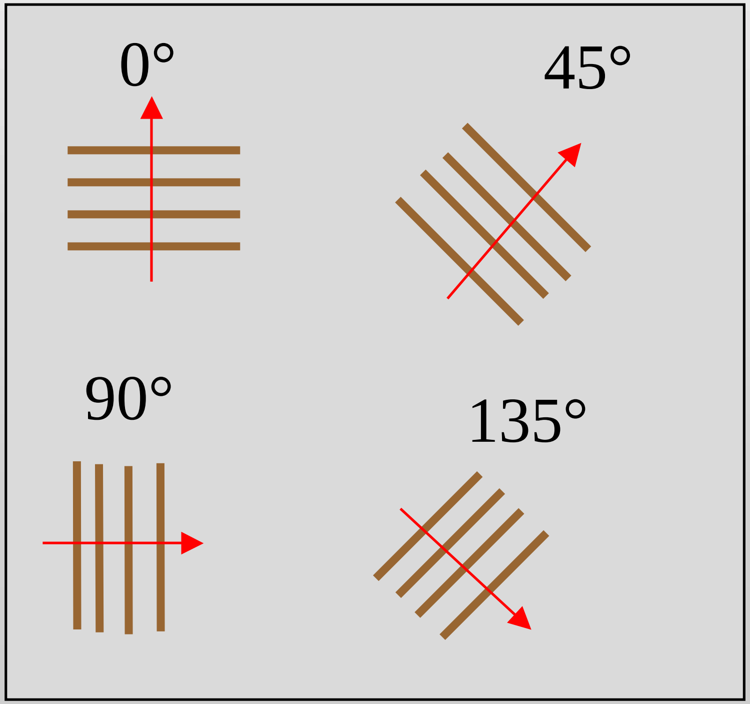
Directional texture

Greyscale image
272x608 pixels

Angle image (size 13x33) calculated with a neighbourhood of size 32x32, moved by 16 pixels







Morphology on angle images

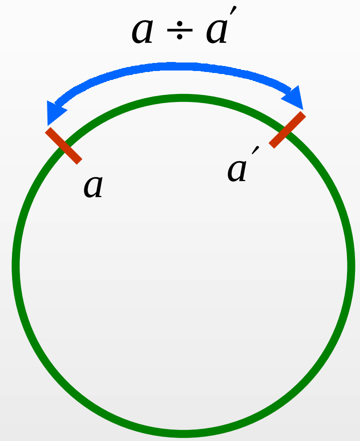
- We would like to use mathematical morphology on these angle images (i.e. with pixels distributed on the unit circle)
- Problem:
  - The unit circle has no order of importance and no dominant position
  - Hence it is impossible to construct a lattice on the unit circle, unless assigning it an arbitrary origin.
- We aim to develop some rotationally invariant morphological operators

Two possible solutions

- Circular centred operators (operators which bring into play a difference)
- Indexed Partitions

Circular centred operators

- Given a unit circle C with centre o
- We choose an arbitrary origin a_0, and indicate the points a_i by their curvilinear coordinates between 0 and 2π from a_0.



- Given two points a and a', the value of the acute angle aoa' is indicated as

$$a \div a' = |\, a - a' \,| \quad \text{if} \quad |\, a - a' \,| \leq \pi$$

$$a \div a' = 2\pi - |\, a - a' \,| \text{if} \quad |\, a - a' \,| \geq \pi$$

- This relation provides a complete ordering of the points on C

$$a_i \quad a_j \quad \text{if} \quad a_i \div a_0 \geq a_j \div a_0$$

$$\text{or if} \quad a_i \div a_0 = a_j \div a_0 \text{ and } a_i - a_0 \leq \pi$$

Gradients (reminder)

- In R^d, to determine the modulus of the gradient, at point x, of a numerically differentiable function f, one uses
  $$2g(x, r) = \vee \{ \, |f(x) - f(y)| \, , \; y \in S(x, r) \, \} \; - \; \wedge \{ \, |f(x) - f(y)| \, , \; y \in S(x, r) \, \}$$
  where $S(x, r)$ is a small sphere centred at x with radius r. The gradient is the limit of g as $r \to 0$.
- In the two-dimensional digital space Z^d, $S(x, r)$ is replaced by the unit square or hexagon $K(x)$

Images with values on C

- $a : E \to C$ is the angle image
- As the definition of the gradient involves only increments, it is transposed to a by replacing $|a(x) - a(y)|$ by $|a(x) \div a(y)|$

$$2(\mathrm{grad}\ a)(x) = \vee \{ \,|\, a(x) \div a(y) \,|, y \in K(x) \} \; - \; \wedge \{ \,|\, a(x) \div a(y) \,|, y \in K(x) \}$$

Example

Hue band

Example

Hue band

Ordinary Hue Gradient

Example

Hue band

Ordinary Hue Gradient

Angular Hue Gradient

Circular-centred top-hat

- Opening by adjunction (erosion, dilation):

  $$\gamma_B(x) = \sup \{ \inf [f(y) , y \in B_i] , i \in I \}$$

  where $\{B_i, i \in I\}$ is the family of structuring elements which contain point x
- The top-hat is therefore:

  $$f(x) - \gamma_B(x) = - \sup \{ \inf [f(y) - f(x), y \in B_i], i \in I \}$$
- As there are only increments of the function f around point x, we can transpose to functions of circular values a as we did for the gradient:

  $$(\text{th } a)(x) = - \sup \{ \inf [- (a(x) \div a(y)) , y \in B_i], i \in I \}$$

Original image

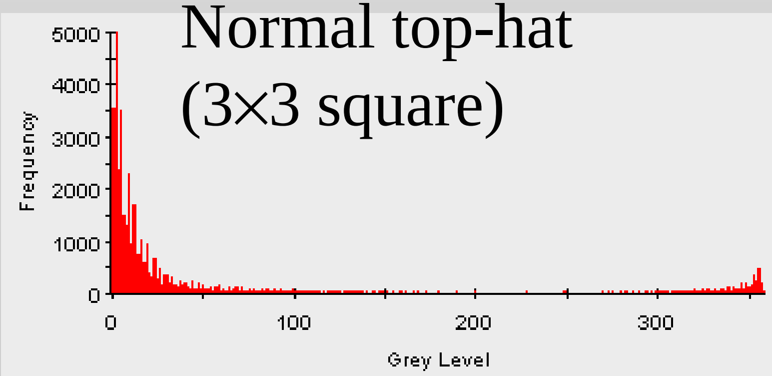
Hue band

Top-hat example

Original image

Hue band

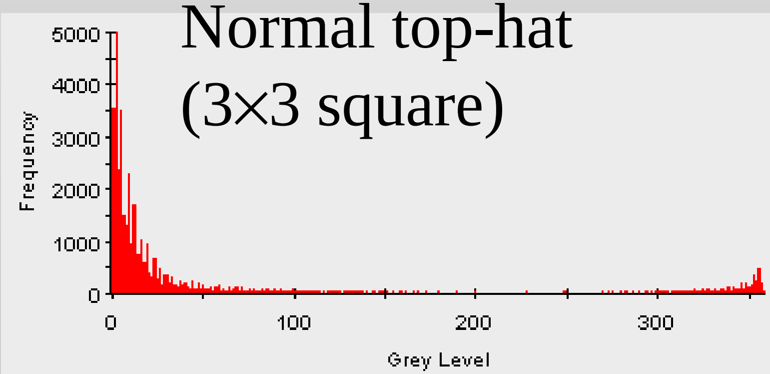
Normal top-hat
(3×3 square)



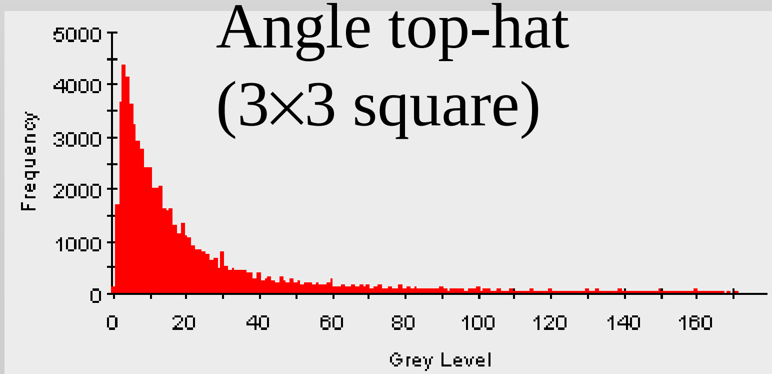
Original image

Hue band

Normal top-hat
(3×3 square)



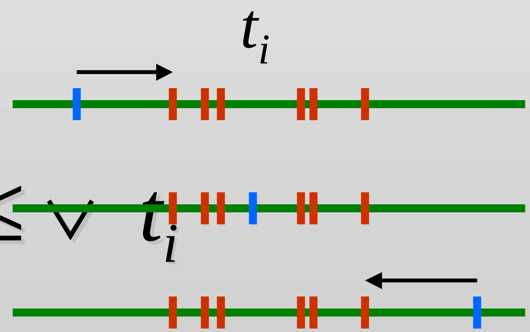
Angle top-hat
(3×3 square)



Morphological centre

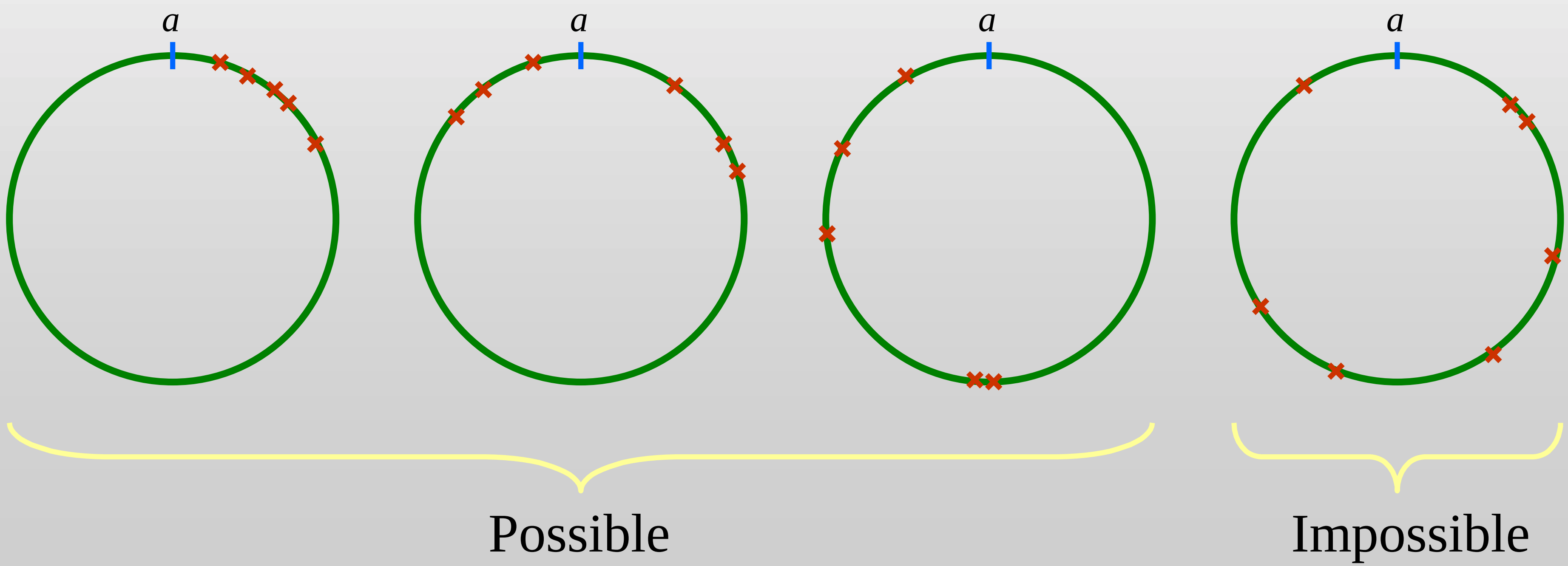
- The classic morphological centre is used if one has n numerical values $t_i \in R$, and a number t which we wish to bring closer to the t_i
- It is defined as

$$\kappa(t) = \wedge\, t_i \quad \text{if} \quad t \le \wedge\, t_i$$
$$\kappa(t) = t \quad \text{if} \quad \wedge\, t_i \le t \le \vee\, t_i$$
$$\kappa(t) = \vee\, t_i \quad \text{if} \quad \vee\, t_i \le t$$



Circular case

- On the circle, it is not always possible to say whether a value *a* is exterior (superior or inferior) to the a_i.
- The following four diagrams illustrate this:



- We use the following definition to exclude the fourth case
  - A family $\{a_i, i \in I\}$ of points on a unit circle are ω-grouped when

$$\vee \ \{ (a_i \div a_j) , \ \ i, j \in I \} \ \ \leq \ \omega \ \leq \ \pi$$

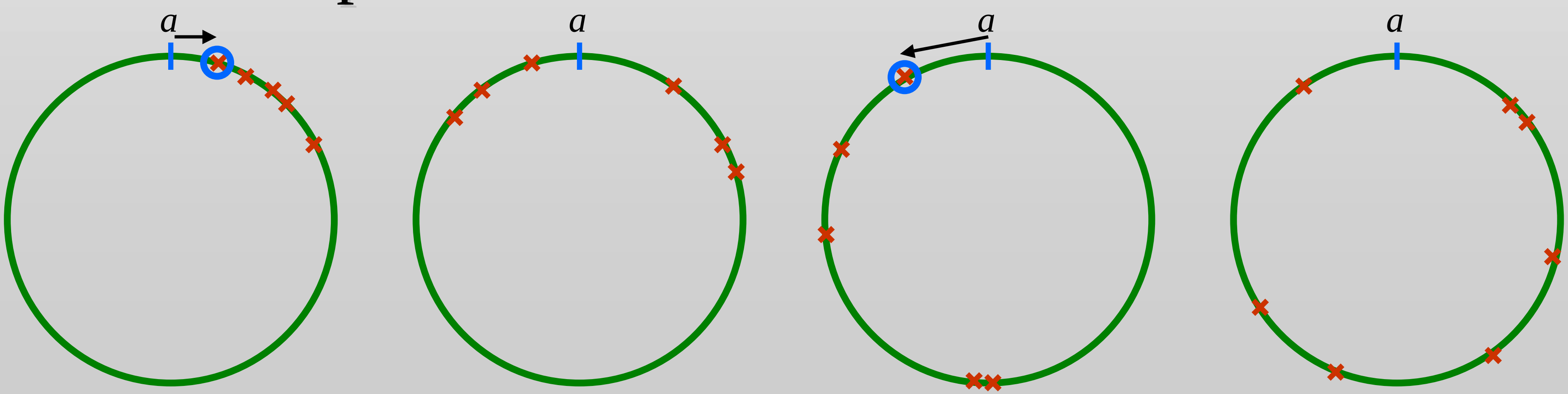
- To characterise a group of points using their coordinates, we use
  - The family $\{a_i, i \in I\}$ of points on a unit circle forms an ω-group if and only if one has

$$\vee \{ a_i , \ i \in I \} \ - \ \wedge \{ a_i , \ i \in I \} \ \leq \ \omega$$

for an arbitrary origin a_0, or for the origin $a_0 + \pi$

Angular morphological centre

- To move a point *a* closer to the points a_i, do the following:
  – If there is an ω-group ($\omega \leq \pi$) and *a* is outside the group, replace *a* by the extremity of the group closest to *a*
  – If there is no group, or if *a* is inside the ω-group, leave it unchanged
- Examples:

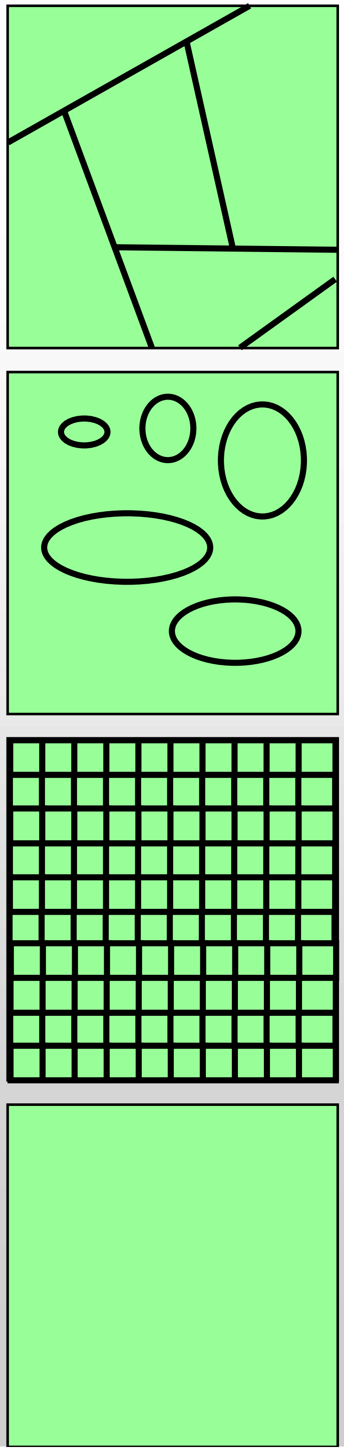


Two possible solutions

- Circular centred operators (operators which bring into play a difference)
- Indexed Partitions

Partitions (reminder)

- E designates the work space
- The set $\Pi(E)$ is provided with a connection X
- We consider the family Δ_0 of partitions of E for which all the classes are connected
- It involves applications $D : E \rightarrow \Pi(E)$ such that for all points x and y in E:
  - $x \in D(x)$ [Every point belongs to a partition]
  - $x \neq y \Rightarrow D(x) = D(y)$ or $D(x) \cup D(y) = \varnothing$ [partitions can’t overlap]
  - $D(x) \in \mathrm{X}$ [the partitions are connected]

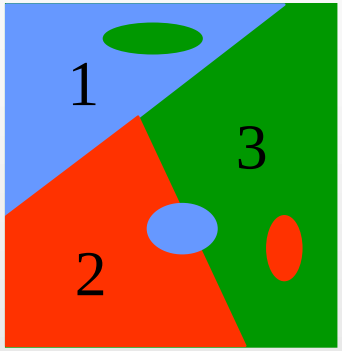


Lattices of partitions

- Given two partitions, not necessarily with connected classes, the inclusion relation
$$D(x) \subseteq D'(x) \quad \text{for all } x \in E$$
defines an order relation, which engenders a lattice
- For partitions of connected classes in Δ_0, this order relation remains valid, but the associated lattice is different
- All families $\{D_i, i \in I\}$ of connected partitions have in Δ_0 a largest minorante D with its class at point x written as
$$D(x) = \gamma_x \ [\cup \ D_i(x) , i \in I \ ]$$
- The largest majorant is the smallest set which is the union of the classes of D_1, and of D_2, …, etc., and which contains point x

Indexed Partitions

- We now limit ourselves to a finite number N of partitions, and associate a label from 1 to N with each partition. These ensembles associated with indices are called phases. The indices are usually associated with some property (colour, direction, etc.)

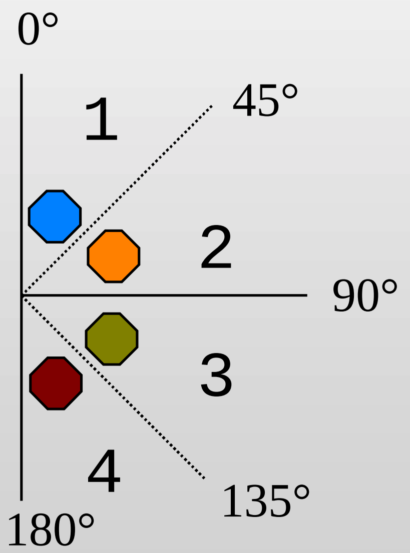


- As there are N phases which fill the space, they are not independent. If we know the first N - 1 phases, the Nth is known
- The ith phase is given by:

$$A_i = \cap \{ D(x, i),\ x \in E \}$$

Creating an indexed partition

- Below is an example of how to convert an angle image (values 0° to 180°) to an indexed partition
  - Decide on a partition size, here 45°
  - Decide on a starting point, here 0°



Partition definition

34	53	15	120
19	165	56	92
49	140	10	120

Original image with pixel values (0° - 180°)

1	2	1	3
1	4	2	3
2	4	1	3

Indexed partition from the image

Lattice of indexed partitions

- **Definition**: An indexed partition on a space *E*, indexed by a finite number *N*, is an application $D : E \rightarrow \Pi(E) \otimes N$ such that the restriction of *D* to $\Pi(E)$ is a connected partition. The *N* sets associated with the gamut of indices (colour, direction, …) are called phases
- Now limit ourselves to *N* - 1 indices. The order relation between two indexed partitions *D* and *D*′ is defined by

$$D \quad D' \Leftrightarrow \begin{cases} D \leq D' & \text{in the sense of connected partitions} \\ A_i \subseteq A_i' & i \in [1, 2, \ldots, N - 1] \end{cases}$$

- The set Δ of partitions with *N* indices is the lattice produced from the *N* lattices associated with the orders above
- This lattice is not unique, because any phase can be chosen to play the role of the *N*th phase

Transformations on Δ

- Let $\psi : D \rightarrow D$ be an increasing operation
- We then have the following relations

$$\{\, A_i \subseteq A'_i \Rightarrow \psi(A_i) \subseteq \psi(A'_i) \,\} \Leftrightarrow \{\, A_i \quad A'_i \Rightarrow \psi(A_i) \quad \psi(A'_i) \,\}$$

$$A_i \subseteq A'_i \text{ for } i \in [1, \ldots, N\text{-}1] \Leftrightarrow A_N \quad A'_N \Rightarrow \psi(A_N) \quad \psi(A'_N)$$

- Consequently, if the operator ψ is increasing for one of the lattices Δ, it is increasing for the others

Cyclic lattices

- The order of increasing operators on Δ is not specified
- When the indices correspond to points on the unit circle, we can associate with them an order of treatment
- The lattice Δ ignores this feature, but the choice of operators acting on it can take this into account
- The term cyclic lattices will be used to mean lattices of indexed partitions with indices on the unit circle

Cyclic operators on indexed partitions Δ

- Two possible approaches:
  - Series operators (e.g.. Closings)
  - Parallel operators (e.g.. Openings)
- By definition, a cyclic operator acting on a cyclic lattice must act systematically on all the indices, either by composition, supremum or infimum

Series Closings

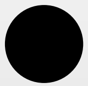
- Let φ_1 be a connected closing on $\Pi(E)$
- Introduce the operator

$$\psi_1\,[D\,(x, 1)] = \gamma_x\,\varphi_1\,(A_1)$$
$$\psi_1\,[D\,(x, i)] = D(x, i) \setminus \gamma_x\,\varphi_1\,(A_1) \quad i = [2,\ldots,N]$$

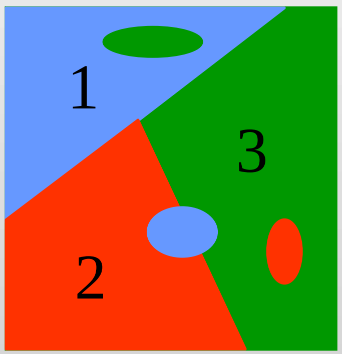
- γ_x is the point connected opening
- The composition $\psi = \psi_N \ldots \psi_2\,\psi_1$, which is a cyclic operator, operates on all the phases. It can be shown that $\psi\,\psi = \psi$ (idempotence) as long as the order of operators is kept the same
- The operator ψ is a cyclic morphological filter on Δ

Illustration of a series closing

$$\psi_3\, \psi_2\, \psi_1\, D$$



Structuring Element



D

Illustration of a series closing

$$\psi_3\ \psi_2\ \psi_1\ D$$

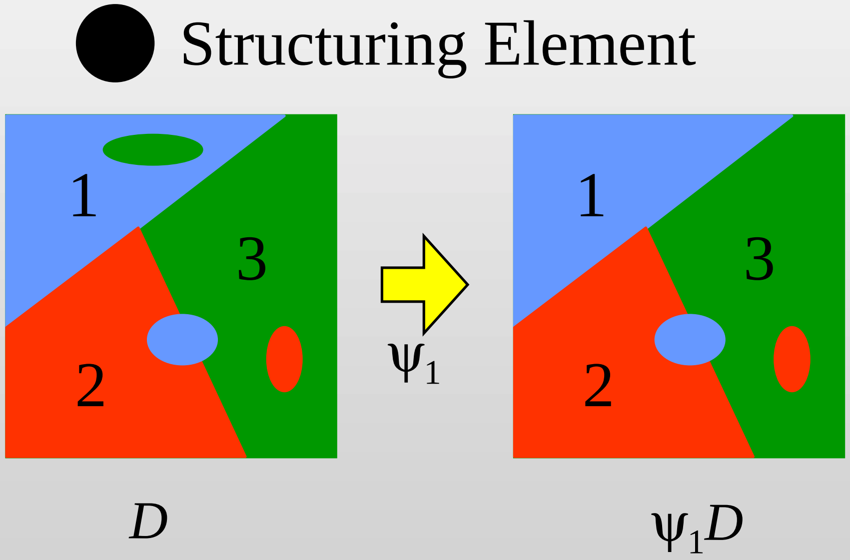


Illustration of a series closing

$$\psi_3\,\psi_2\,\psi_1\,D$$

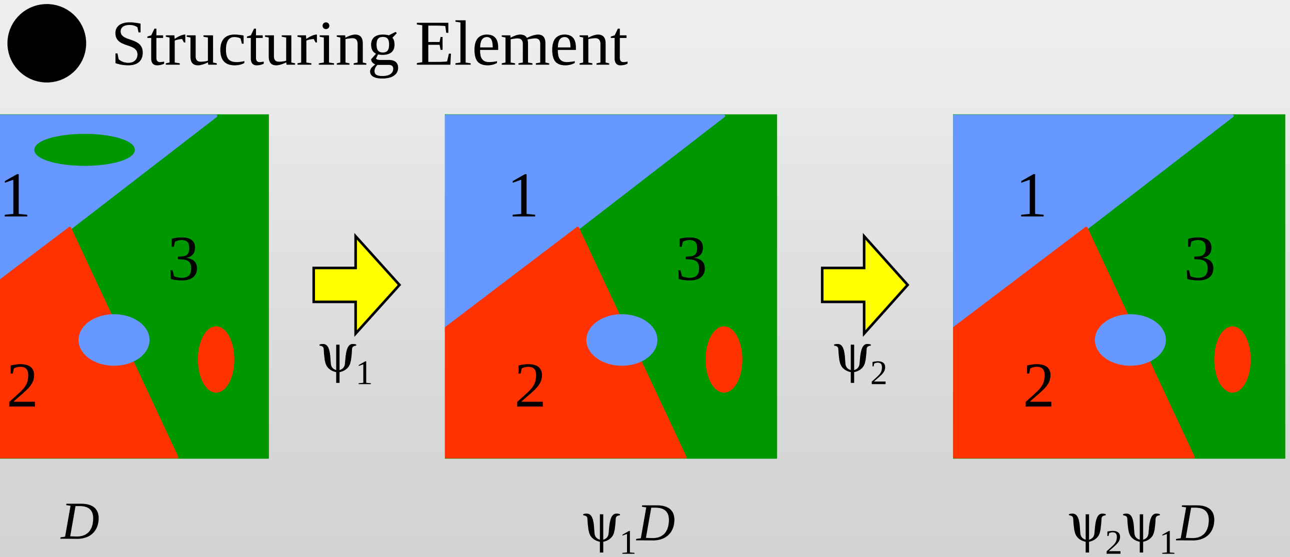
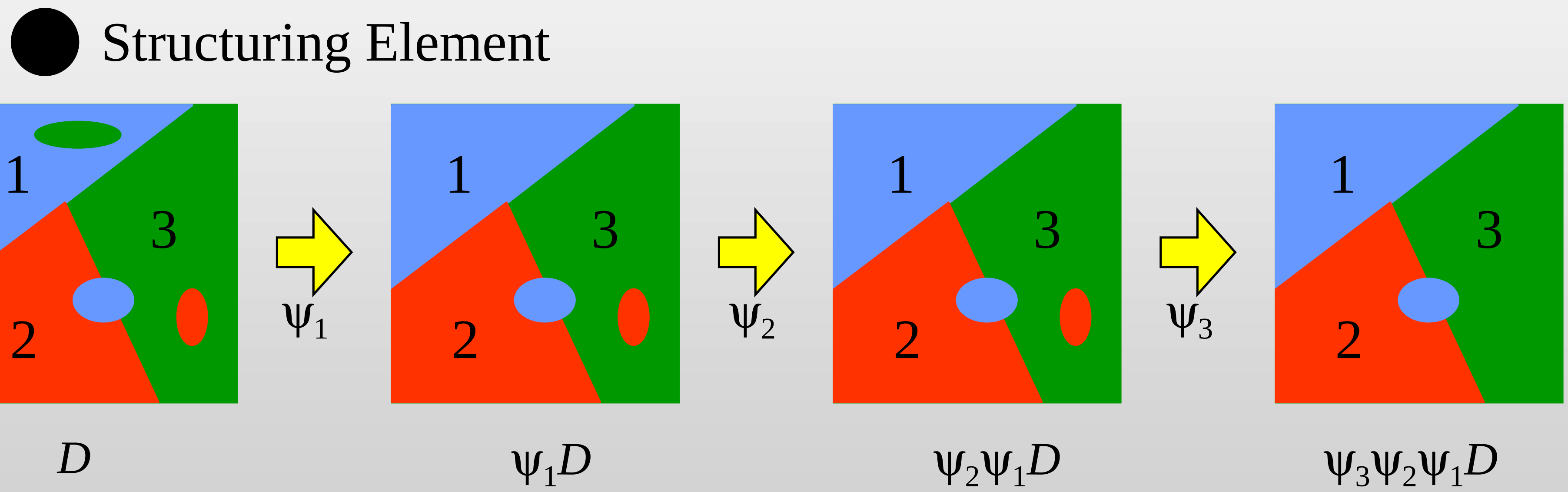
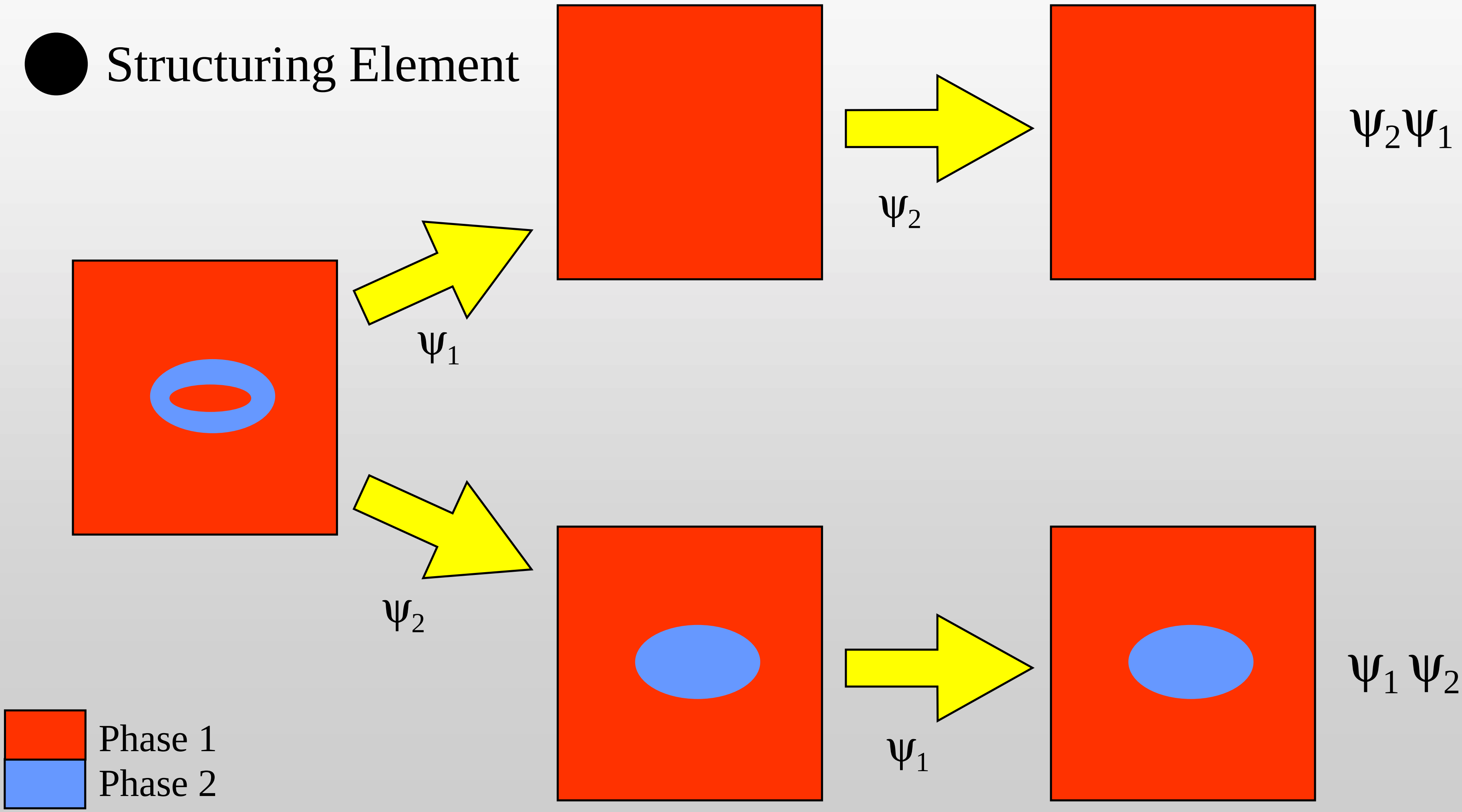


Illustration of a series closing

$$\psi_3\ \psi_2\ \psi_1\ D$$

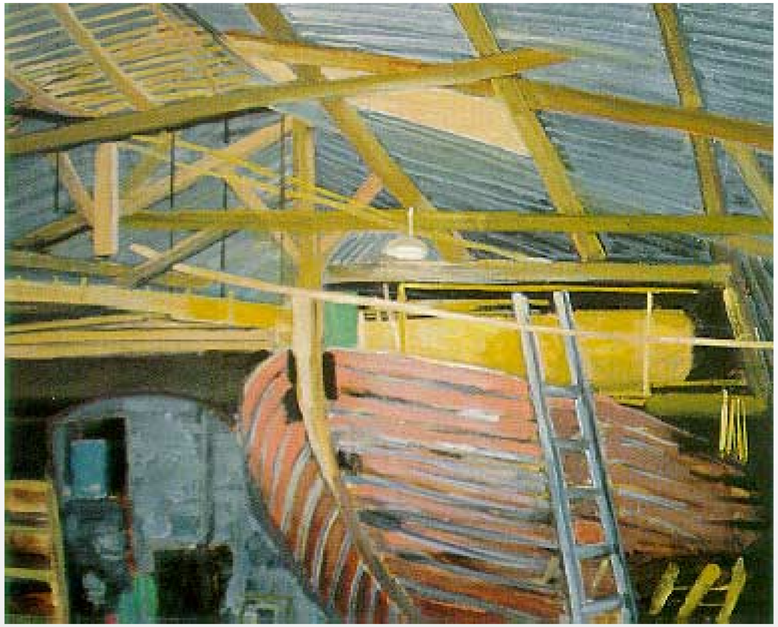


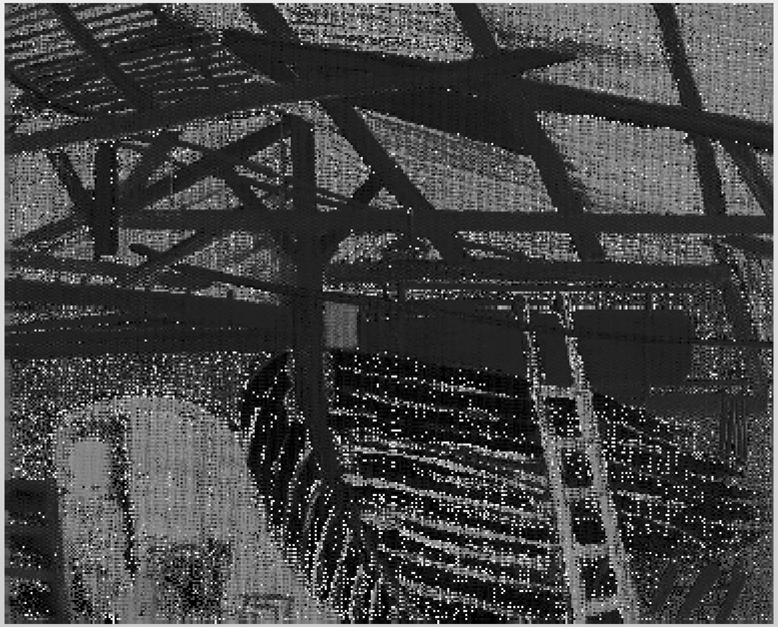
Series Closings are not independent of operator order



Application of the series closing

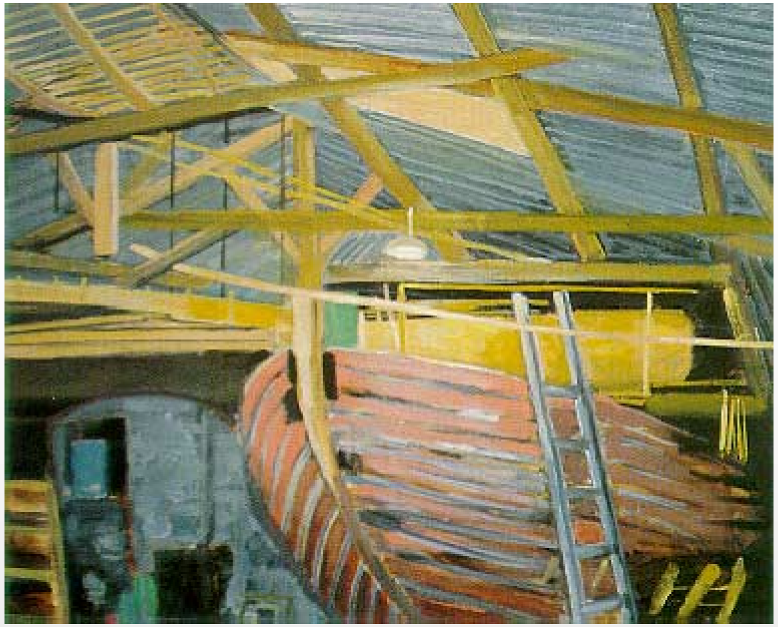
“L’Atelier”, F. Matheron



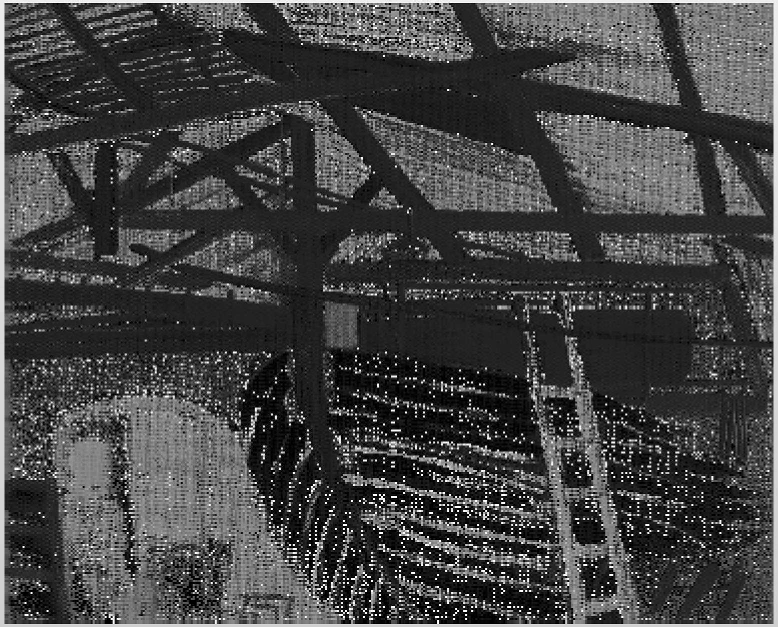


Hue Band

Application of the series closing

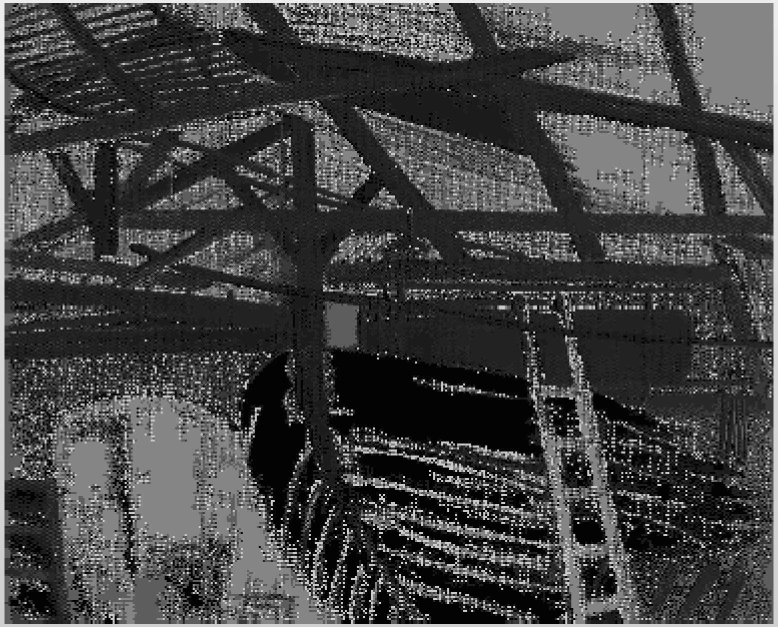


“L’Atelier”, F. Matheron

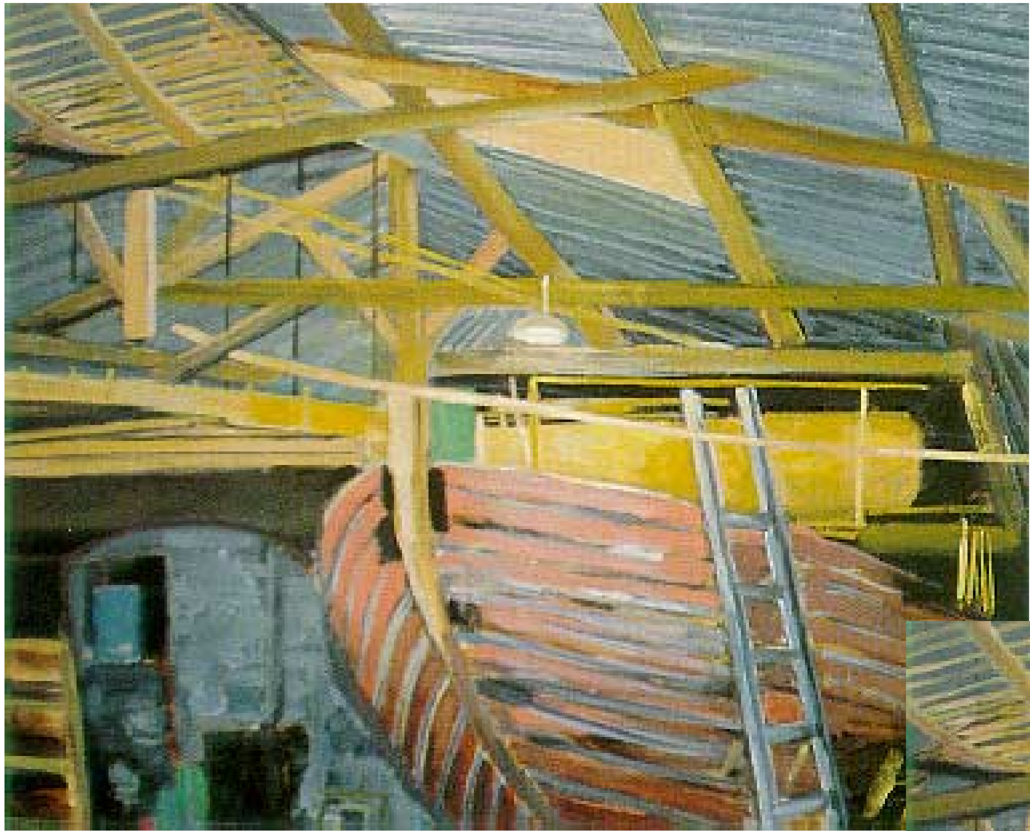


Hue Band

ψ



Reduction to 8 values by histogram equalisation, followed by series closing with a hexagon of size 2



Initial image
(Hue band with
256 grey levels)

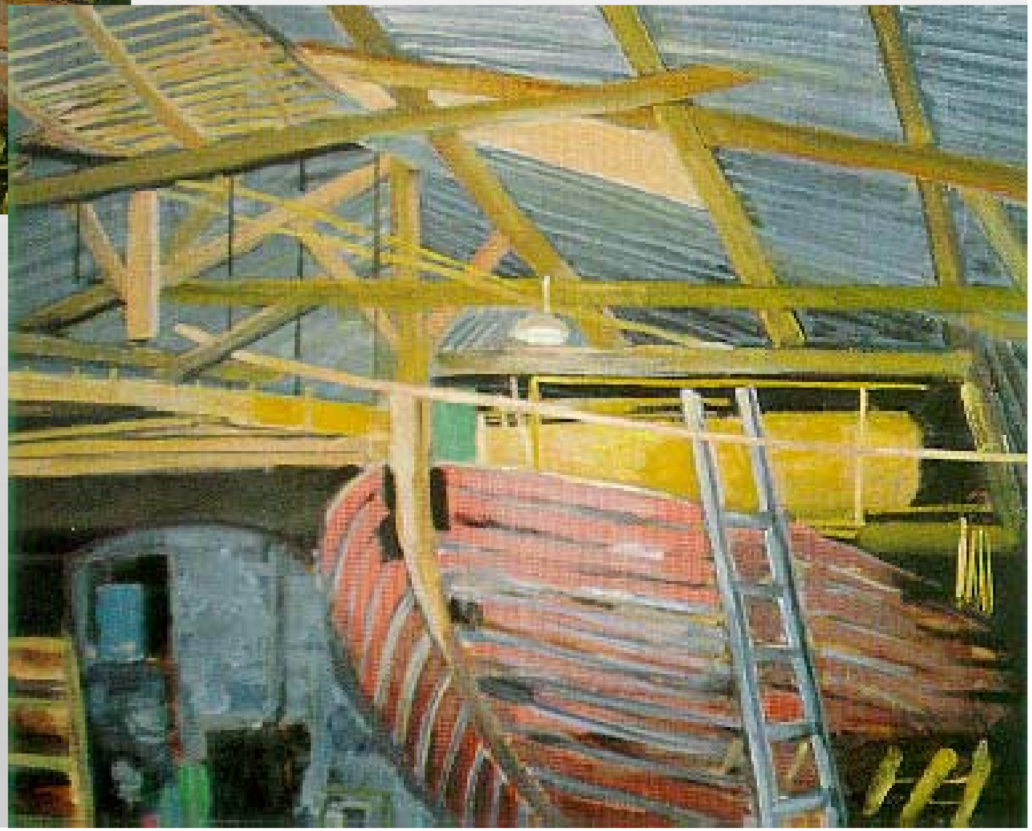


Image with
simplified hue
band
(hue band with
8 grey levels)

Parallel openings

- We now exploit the fact that the *N*th phase has different properties to the others, and use it to indicate residues
- We start with a connected opening $\gamma : \Pi(E) \rightarrow \Pi(E)$, and construct a new partition D^* as

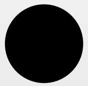
  $D^*_i(x) = \gamma_x [\gamma(A_i)] = \gamma [D_i(x)]$

  if $\gamma[D_i(x)] \neq \emptyset$, $i \, \varepsilon \, [1, \ldots\ldots, N\text{-}1]$

  $D^*_N(x) = \gamma_x (A_N)$ where $A_N = \{ x : \gamma_x [\gamma(A_i)] = \emptyset$, $i \, \varepsilon \, [1, \ldots\ldots, N\text{-}1] \}$
- We denote as $\gamma^* : \Delta \rightarrow \Delta$ the operator which transforms D into D^*
- γ^* is a morphological filter on Δ and an opening for the $N - 1$ phases A_i
- We privilege its action on the phases, and call it a X-opening

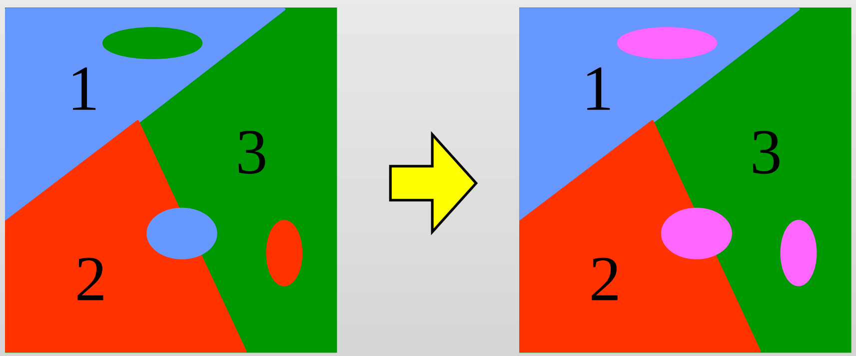
Illustration of a parallel opening

- Parallel as all phases are changed together



Structuring Element

$N = 4$



D *D**

The phase A_4 is indicated by the colour

Circular parallel opening

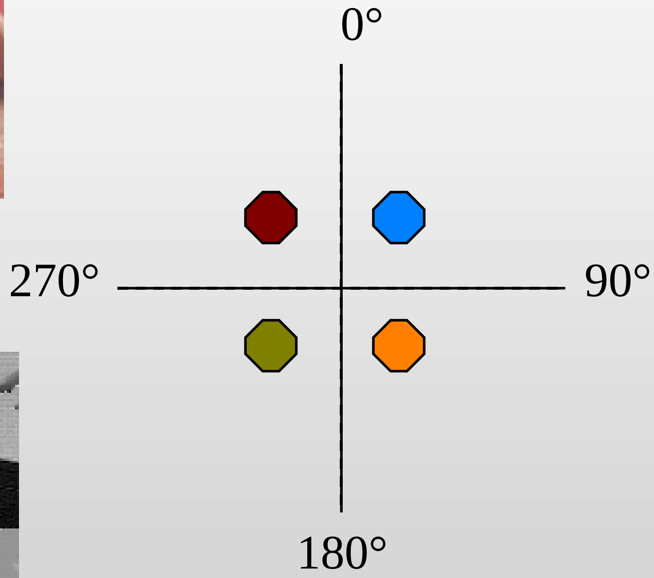
- Divide the circle into N - 1 sectors of size $\omega = 2\pi / (N{-}1)$ starting from an angular origin α.
- The result is a partition of E into N - 1 phases A_i, and by application of γ^*, an Nth phase $A_N(\alpha)$.
- The phase $A_N(\alpha)$ depends on the origin, so we isotropise it by intersection

$$A_N = \cup\{A_N(\alpha), 0 \leq \alpha \leq 2\pi\}$$

- A point belongs to A_N only if it disappears from every opening for all values of α
- A_N can be interpreted as the result of a very simple isotropic closing

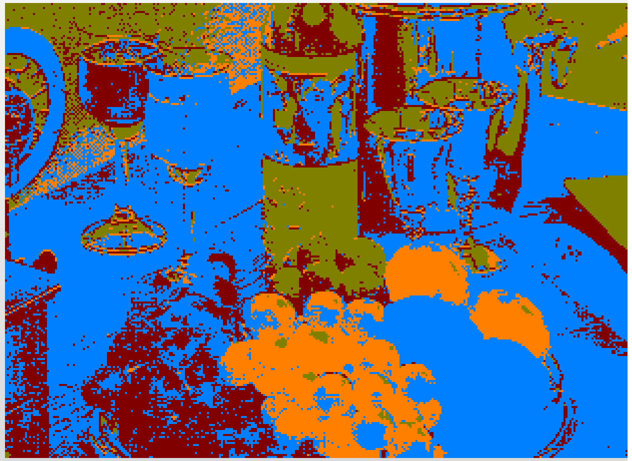
Original Image

Hue Image



Class definition

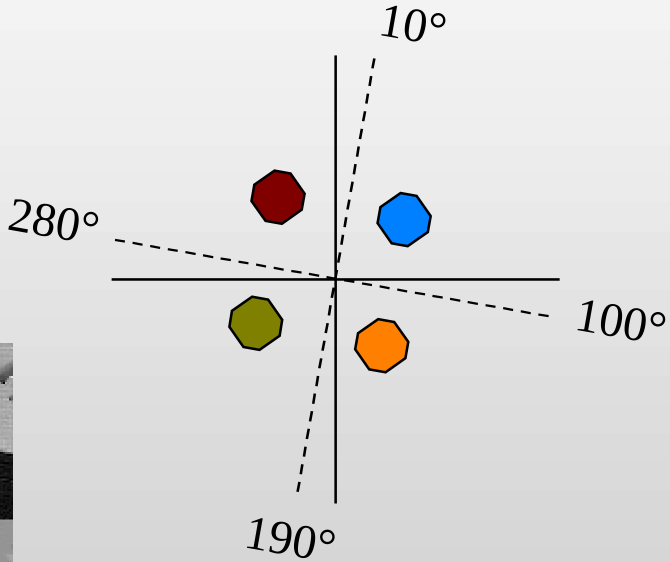
$\alpha = 0°, \omega = 90°$

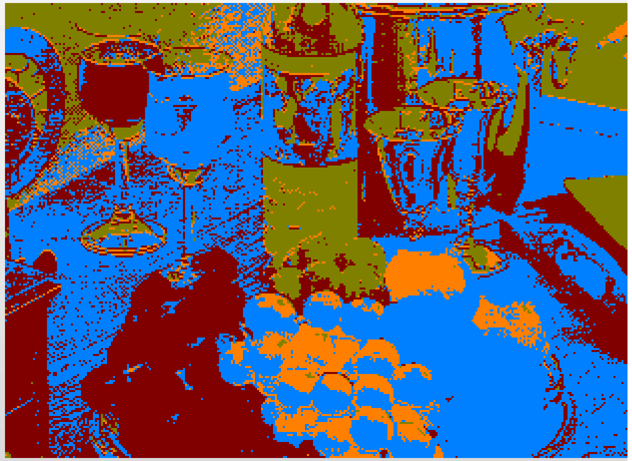


Labelled Image

Original Image

Hue Image





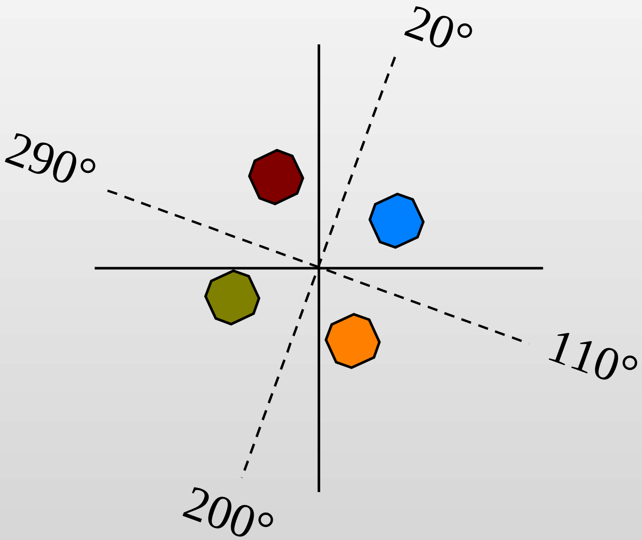
Labelled Image

Class definition

$\alpha = 10°, \omega = 90°$

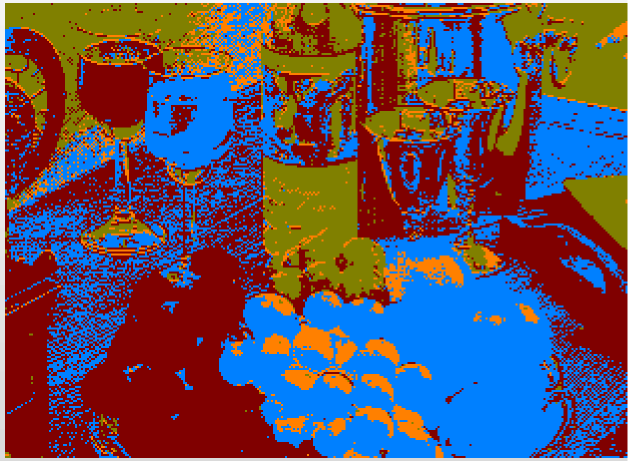
Original Image

Hue Image



Class definition

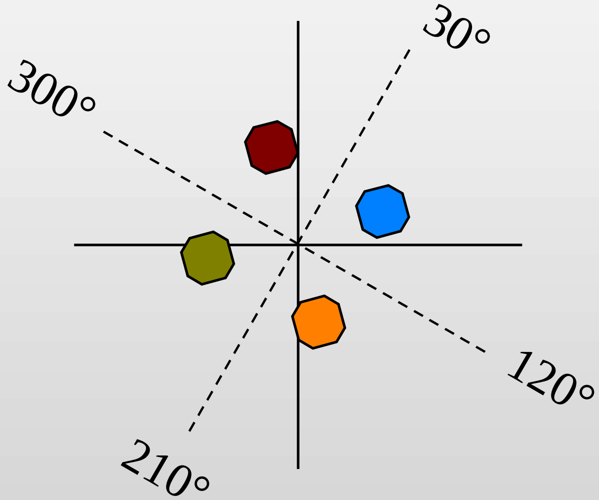
$\alpha = 20°, \omega = 90°$



Labelled Image

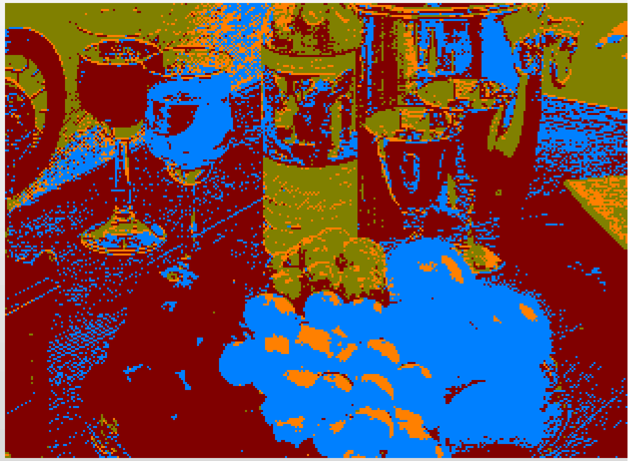
Original Image

Hue Image



Class definition

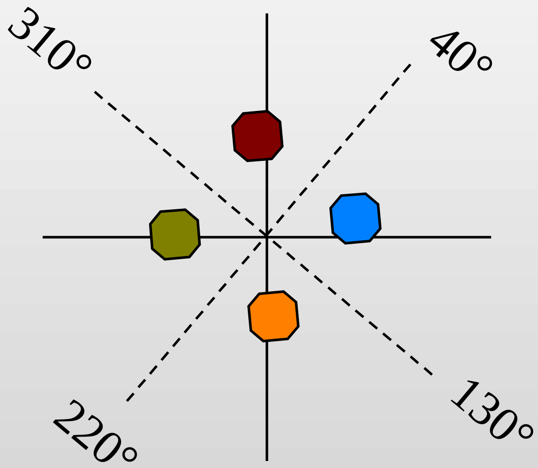
$\alpha = 30°, \omega = 90°$



Labelled Image

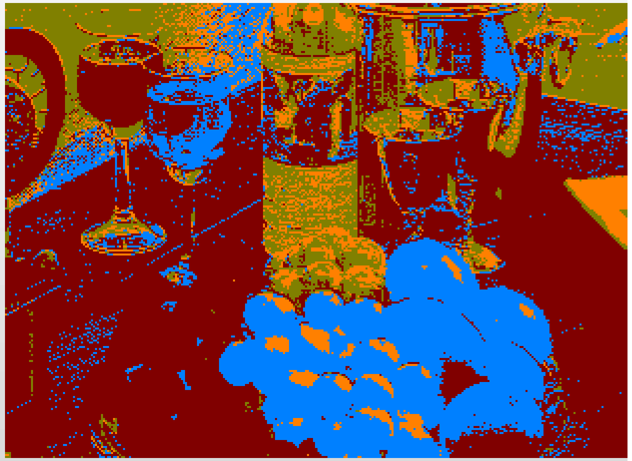
Original Image

Hue Image



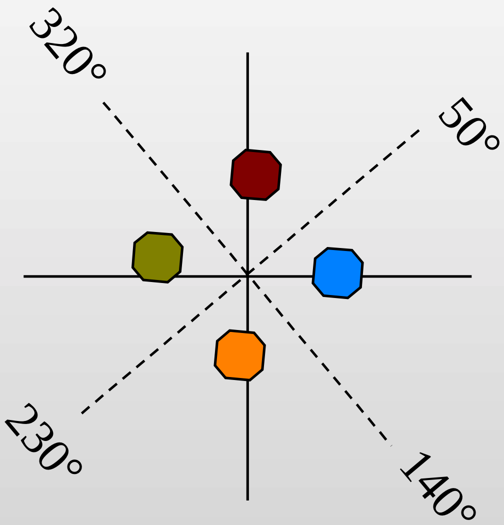
Class definition

$\alpha = 40°, \omega = 90°$



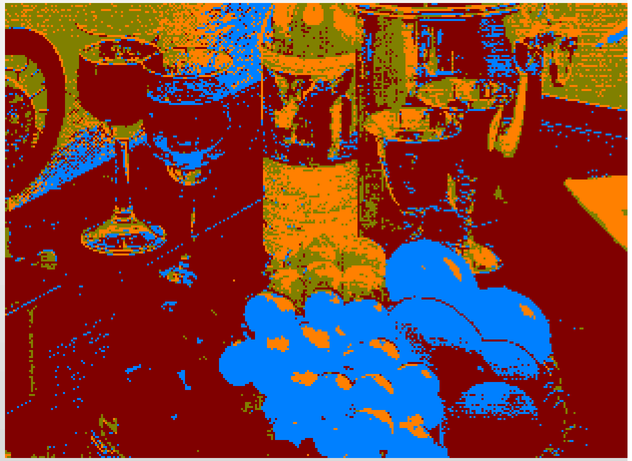
Labelled Image

Original Image



Class definition

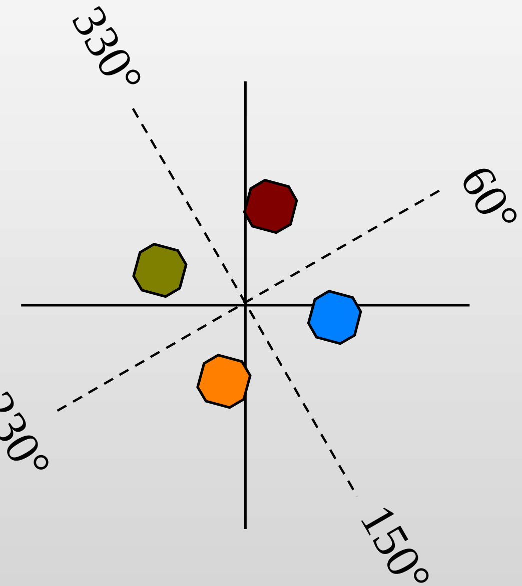
$\alpha = 50°, \omega = 90°$

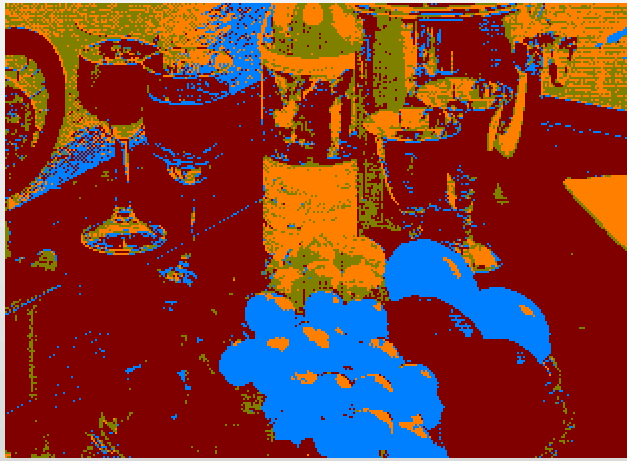


Labelled Image

Hue Image

Original Image





Labelled Image

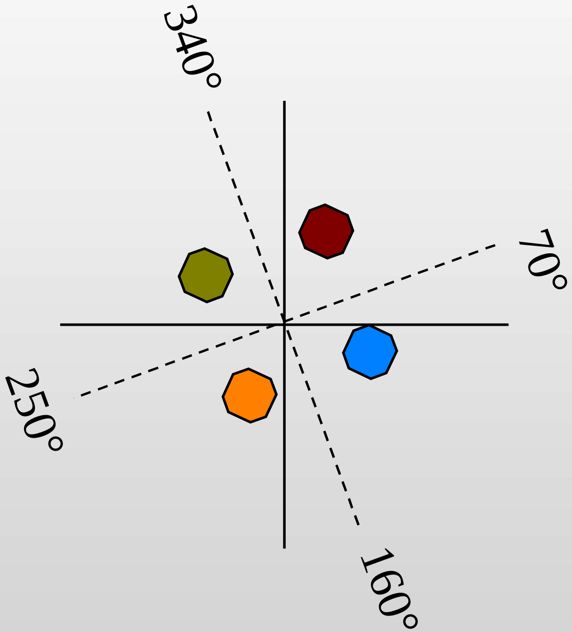
Hue Image

Class definition

$\alpha = 60°, \omega = 90°$

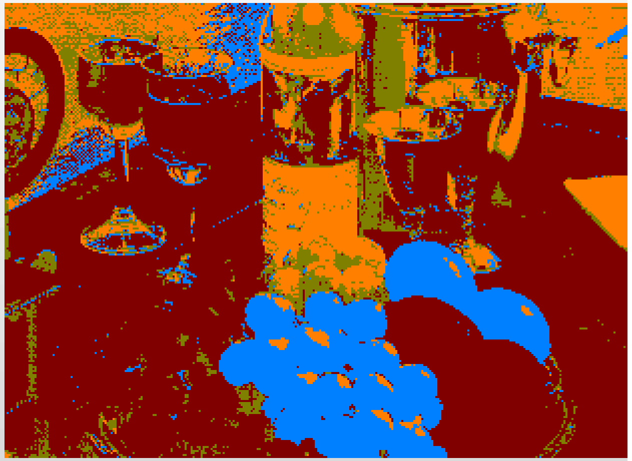
Original Image

Hue Image



Class definition

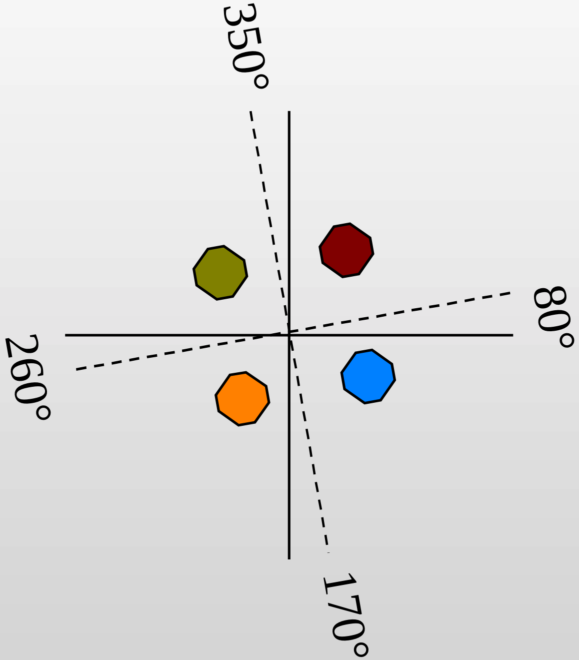
$\alpha = 70°, \omega = 90°$



Labelled Image

Original Image

Hue Image

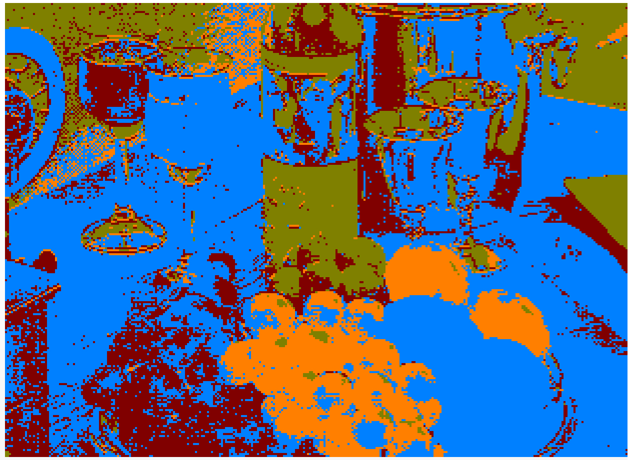


Class definition

$\alpha = 80°, \omega = 90°$

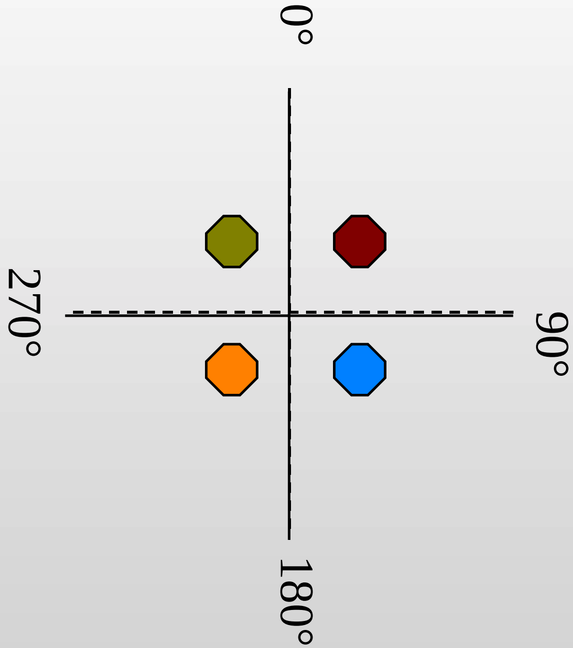
Labelled Image

$\alpha = 0°, \omega = 90°$ →



Original Image

Hue Image



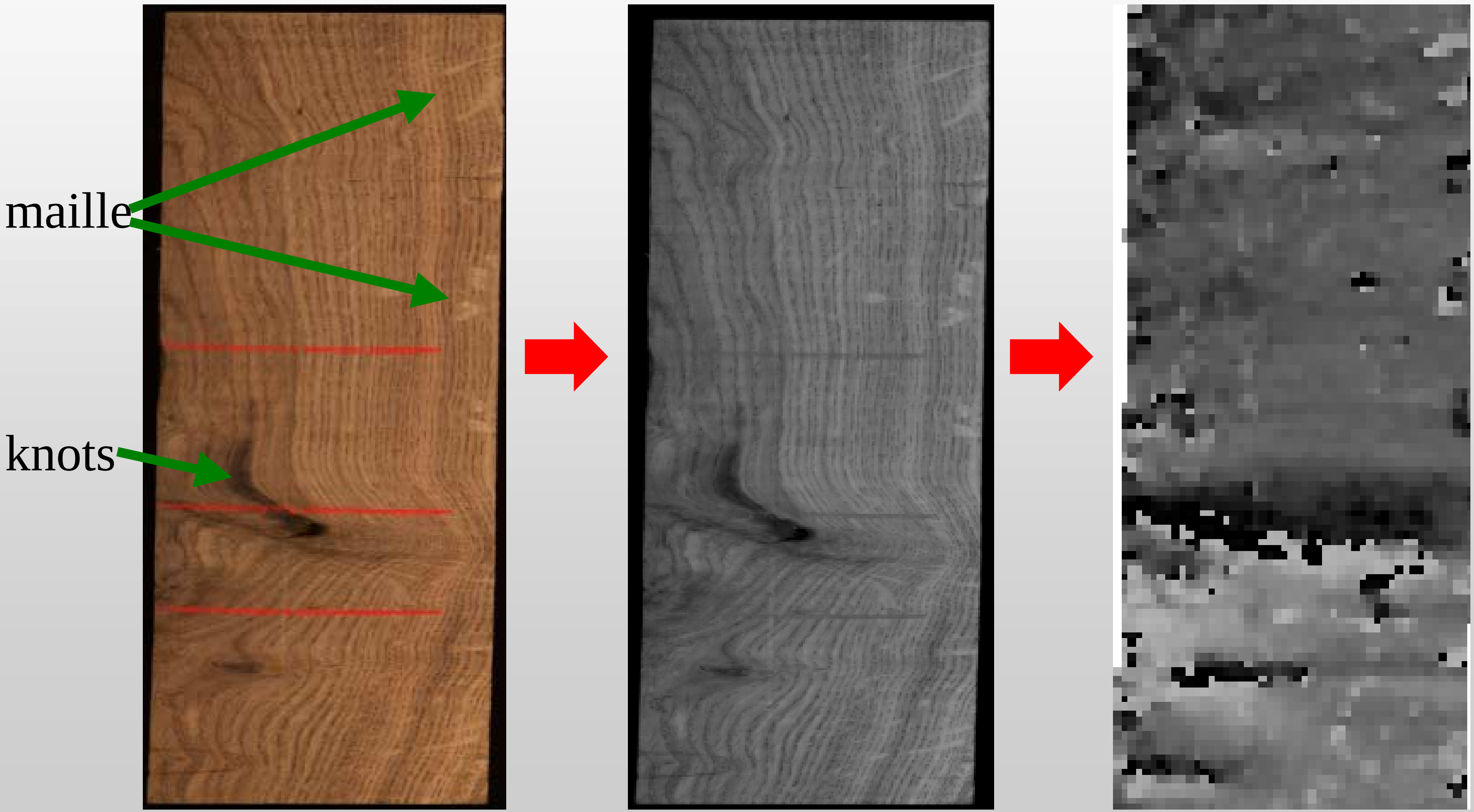
Class definition

$\alpha = 90°, \omega = 90°$

Labelled Image

Example of application to defect detection

Initial image | Luminance image | Reduced angle image



Characteristics of the knots and the maille

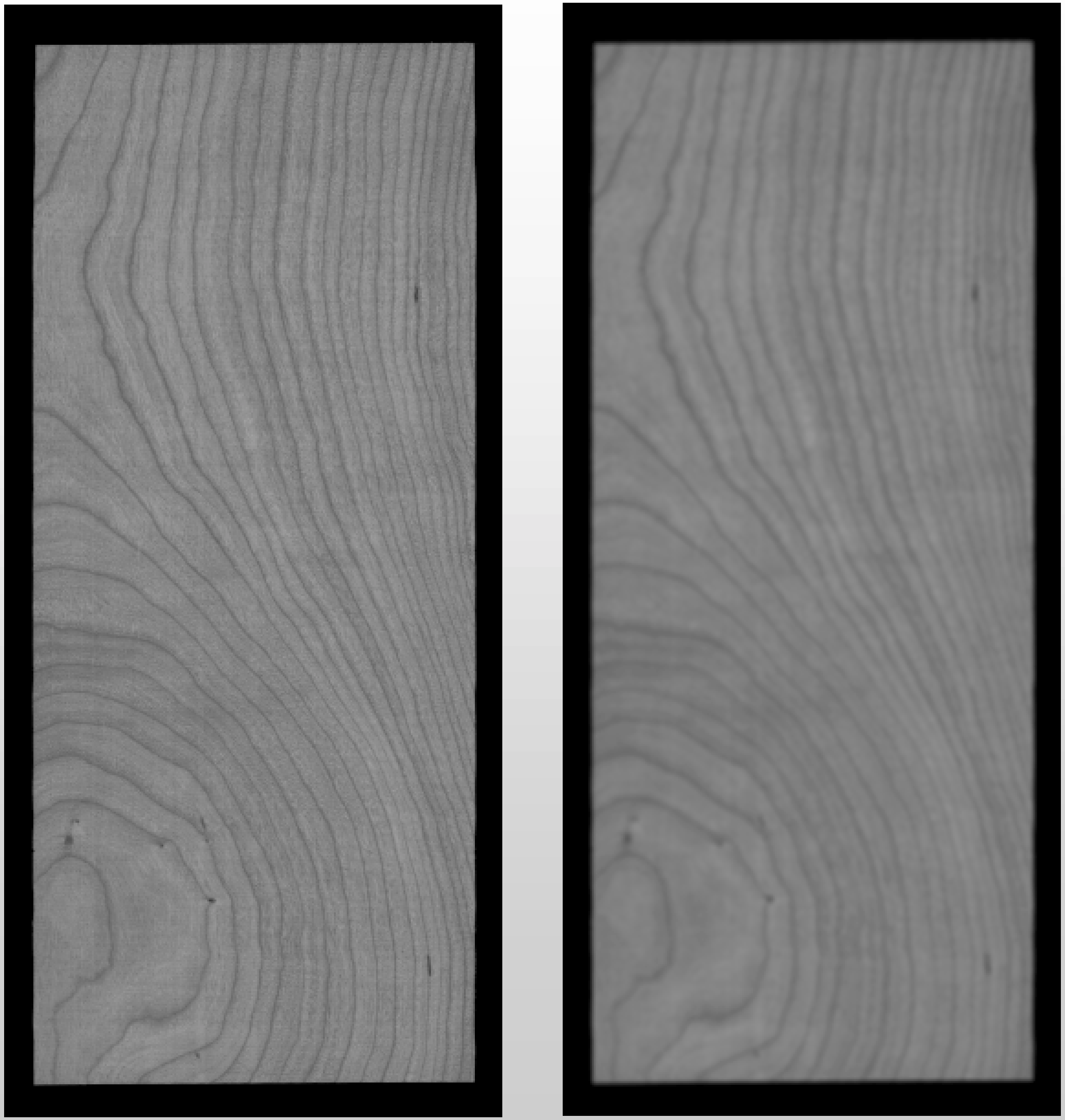
- Colour
  - In general, the knots are very dark
  - The maille is light, but the same colour as other light regions of the wood
- Texture
  - There is a strong perturbation in the grain direction near knots
  - The maille cuts the grain lines, thereby producing a slight modification of the dominant direction

The Rao algorithm

- Gaussian filter
- Calculation of the horizontal and vertical gradients of the smoothed images
- Calculation of an angle at each pixel from images of the horizontal and vertical gradient
- The dominant angle is calculated in the neighbourhoods to produce an angle image
- Each pixel in the angle image corresponds to the dominant angle in a neighbourhood

Chain of treatment

- Initial Image and Gaussian smoothed image (5x5 filter)

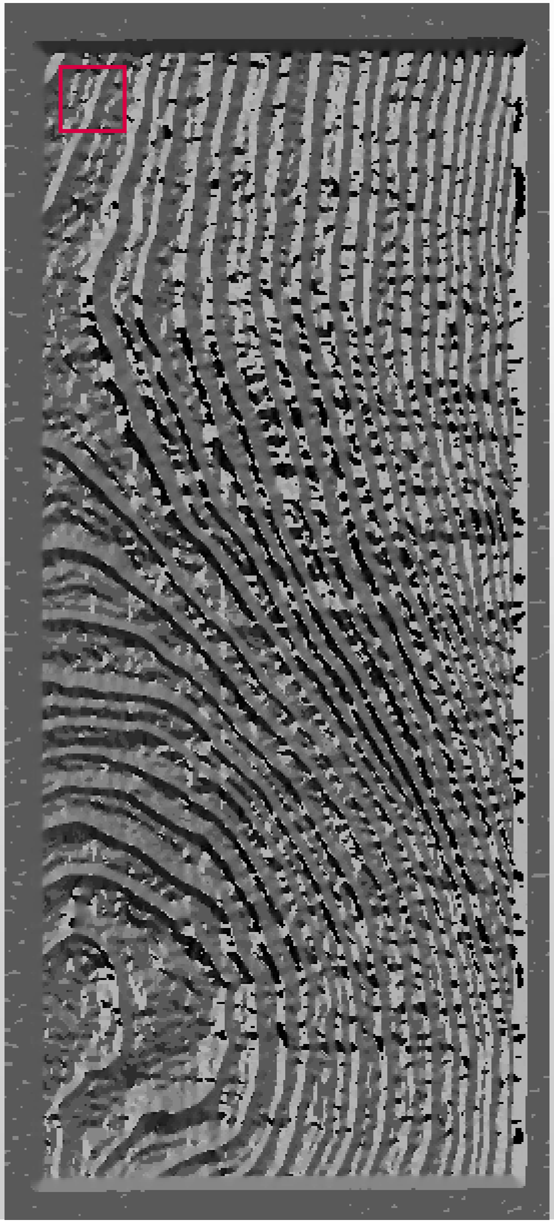


- The gradient images of the Gaussian smoothed image

Vertical edges

Horizontal edges

■ Magnitude and initial angle images



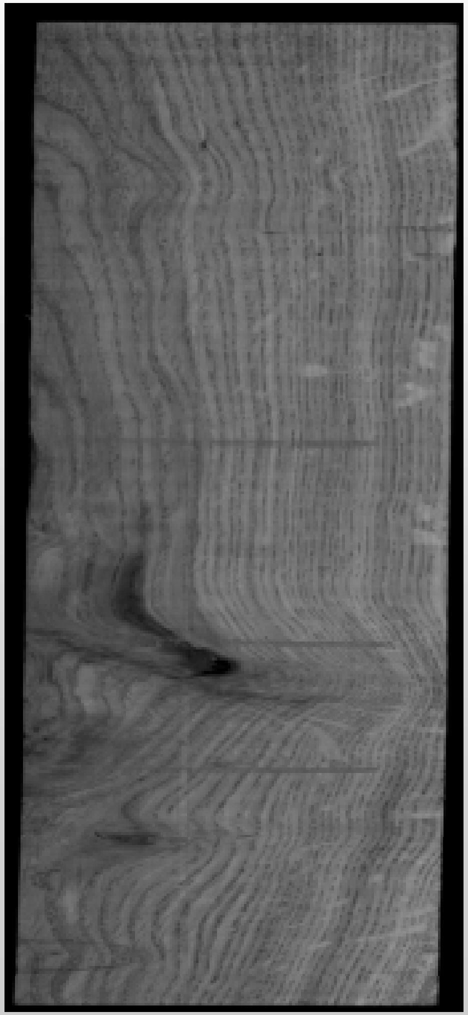
In each 32x32 frame, the dominant direction (between 0° and 180°) is calculated using

$$\tan 2\theta = \frac{\sum_{j=1}^{N} R_j^2 \sin 2\theta_j}{\sum_{j=1}^{N} R_j^2 \cos 2\theta_j}$$

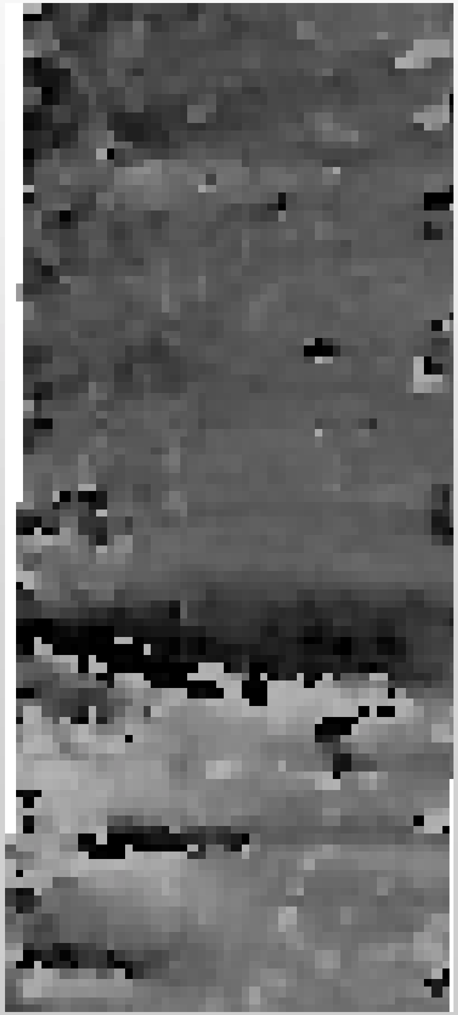
where R_j is the magnitude and θ_j is the angle at point j

Final Result

Luminance image

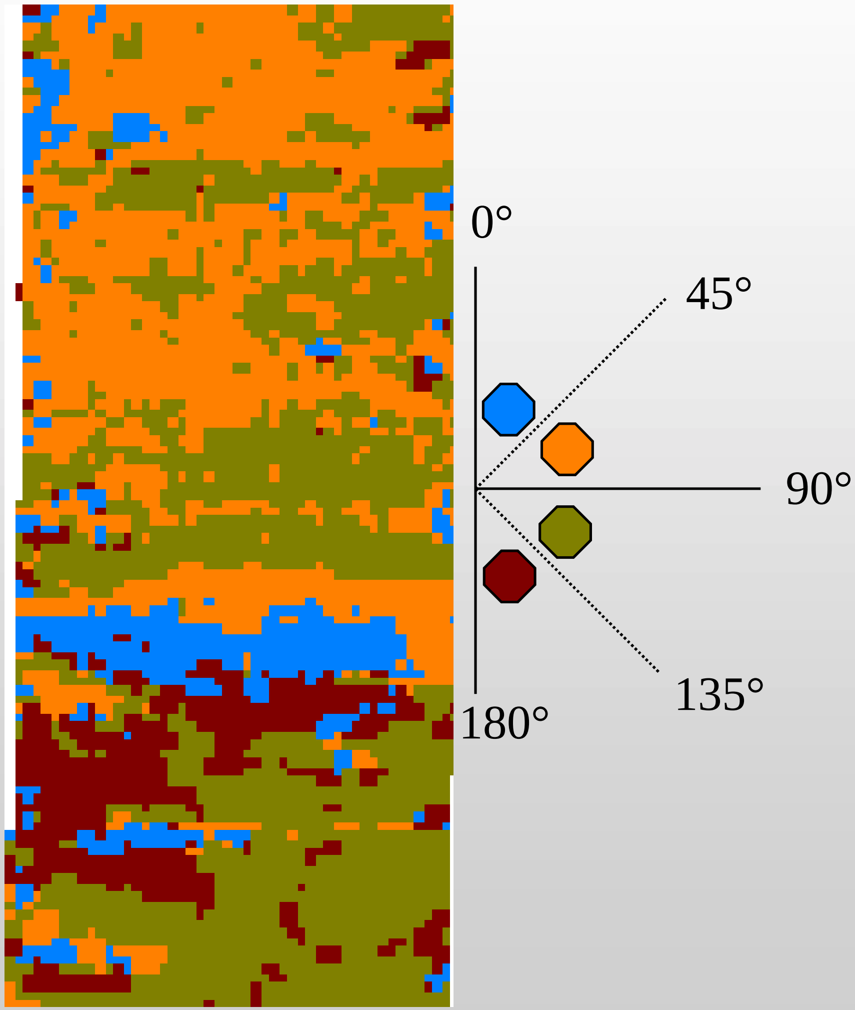


Reduced angle image



- Two labelisations with different α to simulate the rotation of α

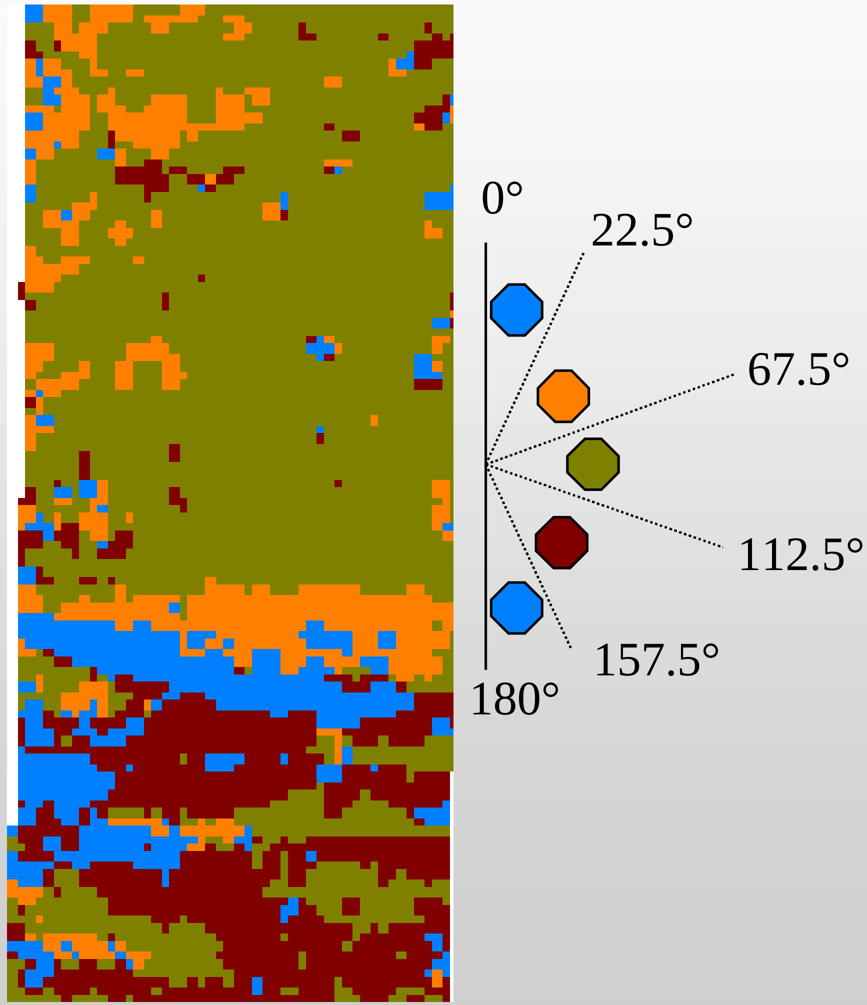
1



$\alpha = 0°, \omega = 45°$

$N = 5$

2

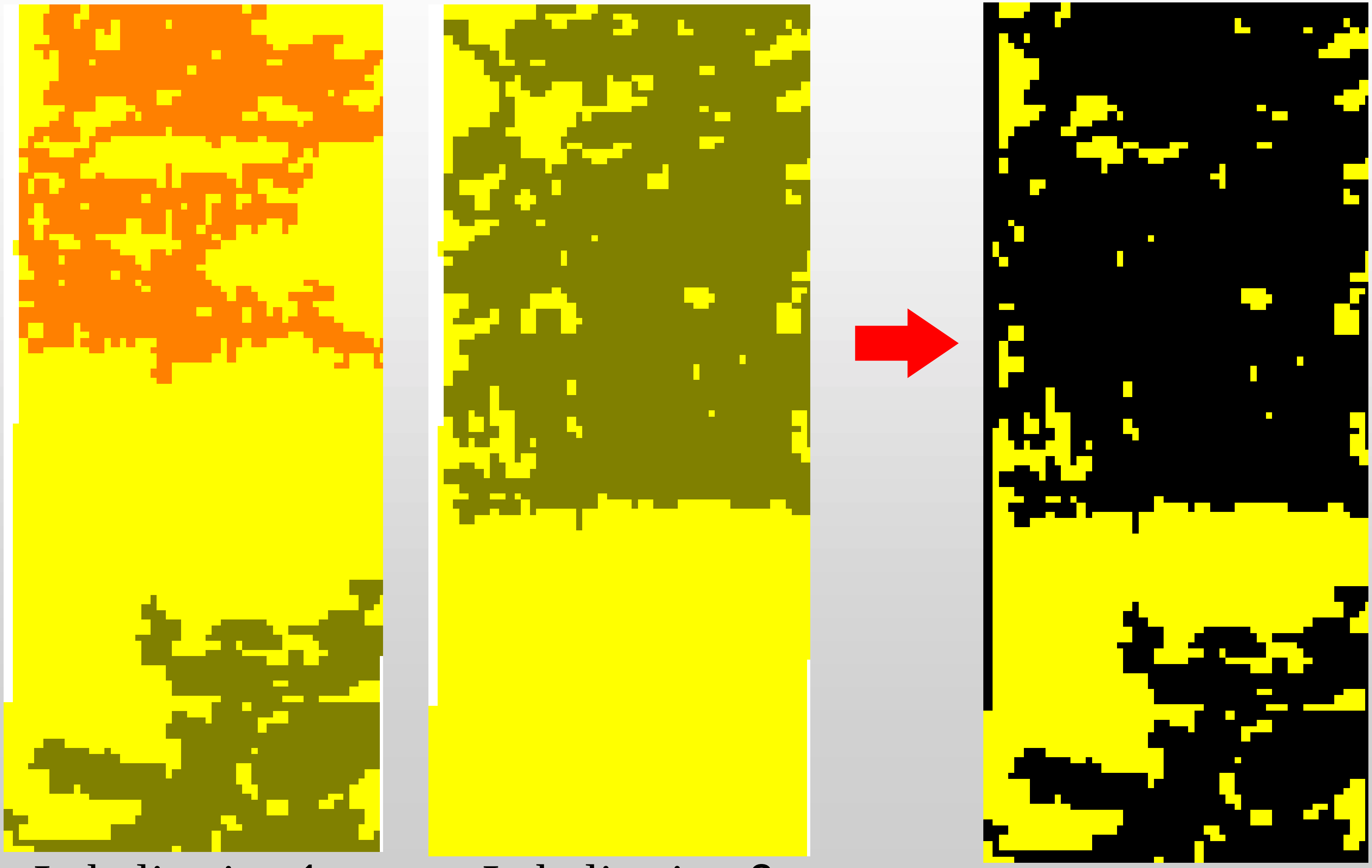


$\alpha = 22.5°, \omega = 45°$

■ Cyclic opening of size 9x9

Residue (phase 5) indicated in ⬡

Intersection of the residues

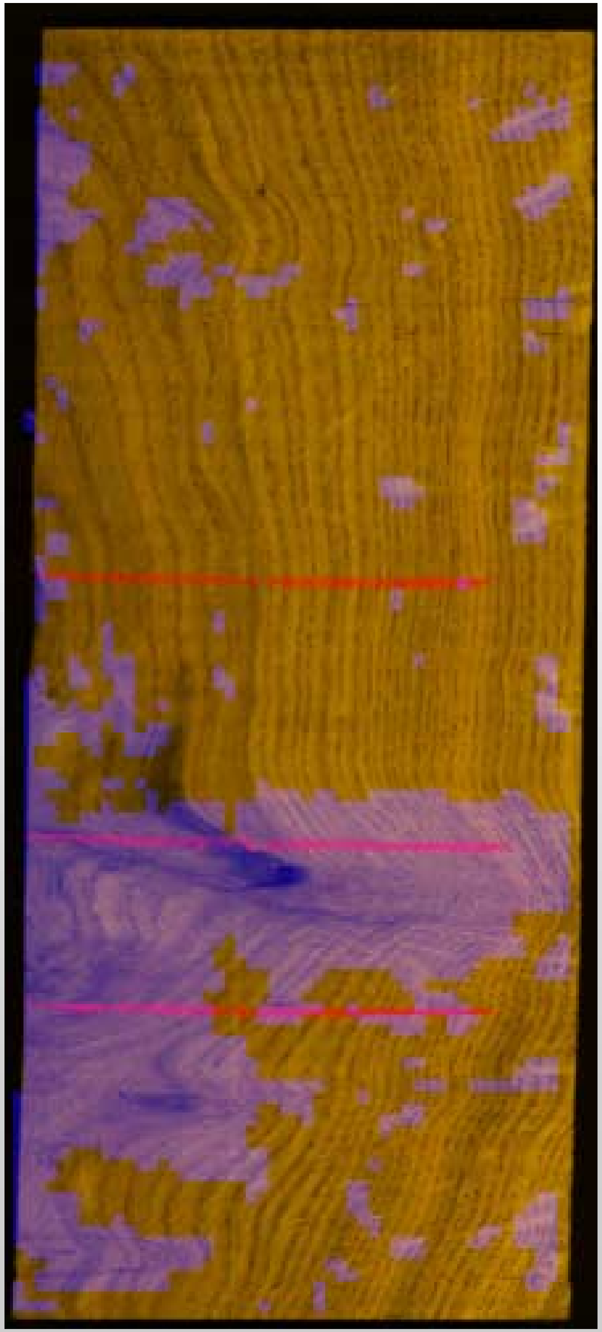


Labelisation 1

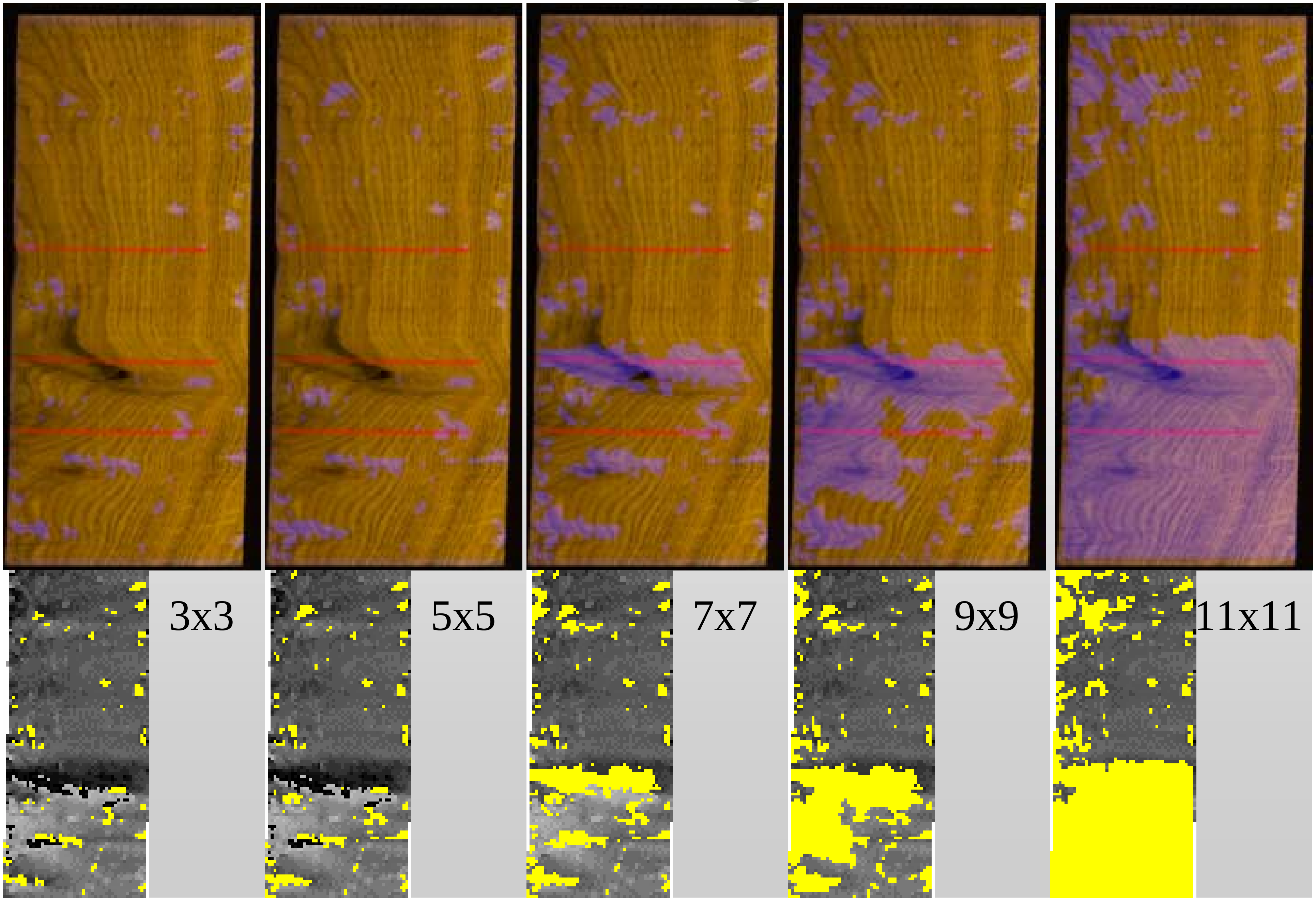
Labelisation 2

Detection of knots and maille

- Projection of the residues found onto the original image

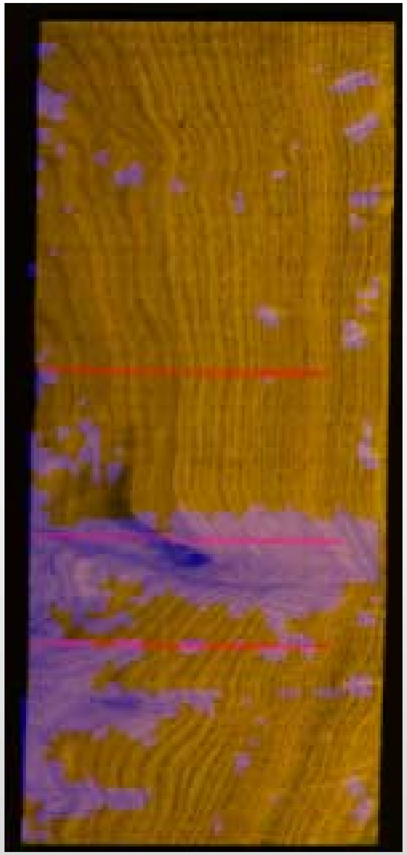


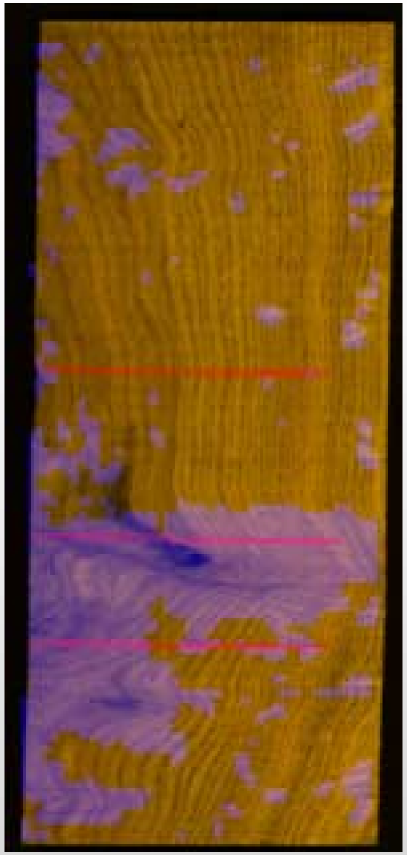
Effect of structuring element size

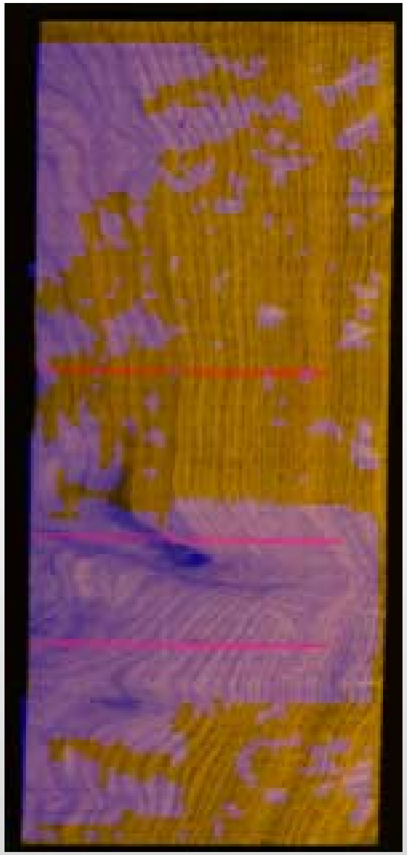


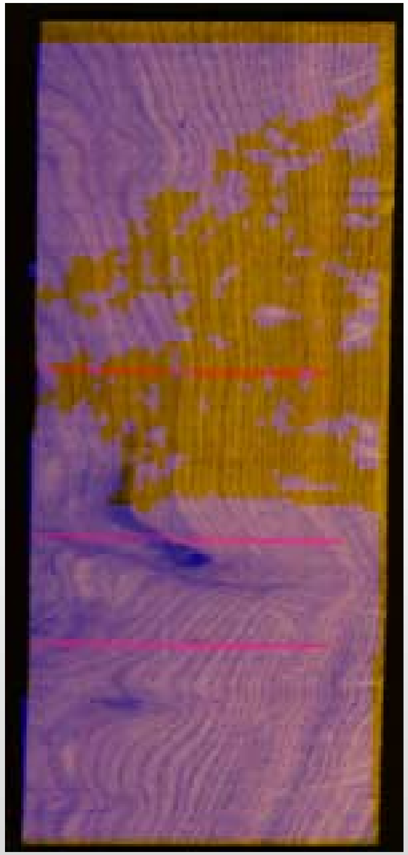
Change in partition definition

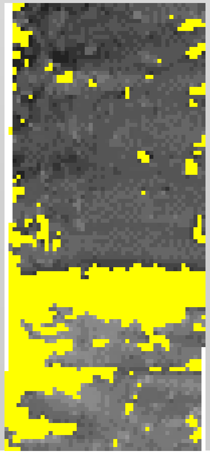
9x9 opening





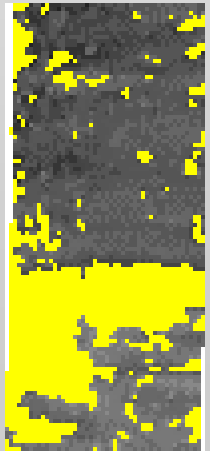






ω=60°,

α=0° and α=30°



ω=45°,

α=0° and α=23°

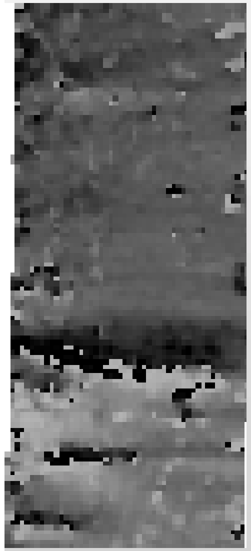
ω=30°,

α=0° and α=15°

ω=20°,

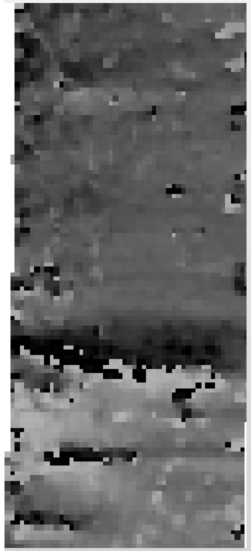
α=0° and α=10°

Use of circular-centred top-hat



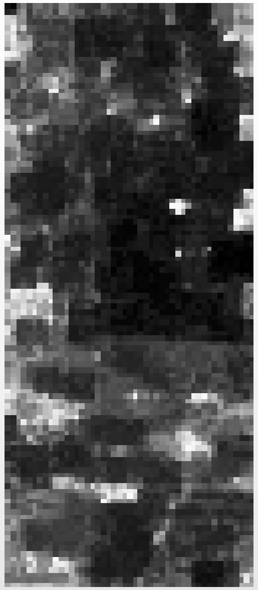
Angle
image

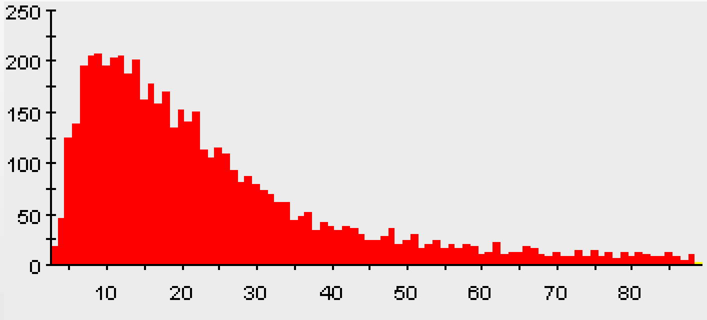
Use of circular-centred top-hat



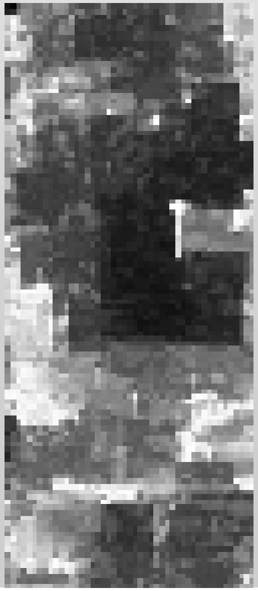
Angle image

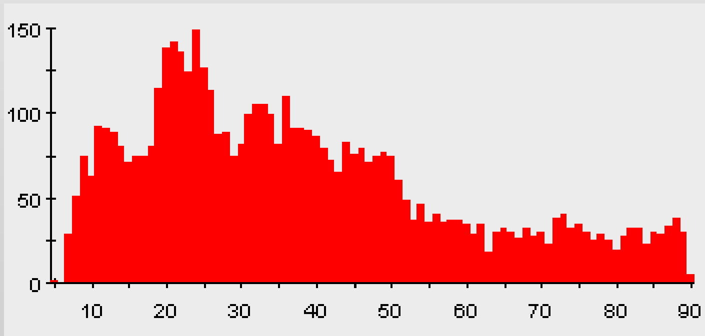
5x5





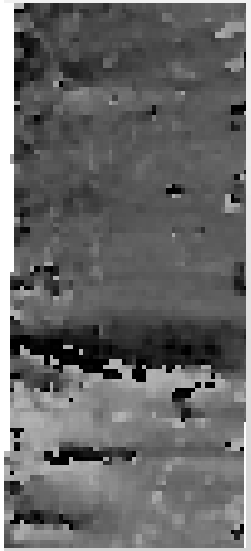
Angular top-hat and histogram





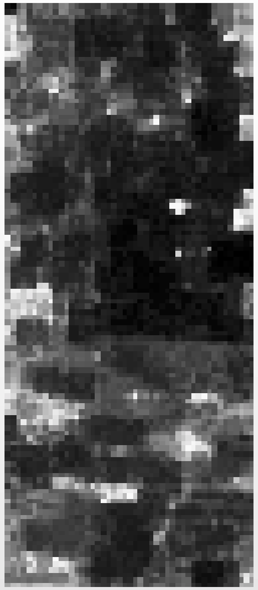
9x9

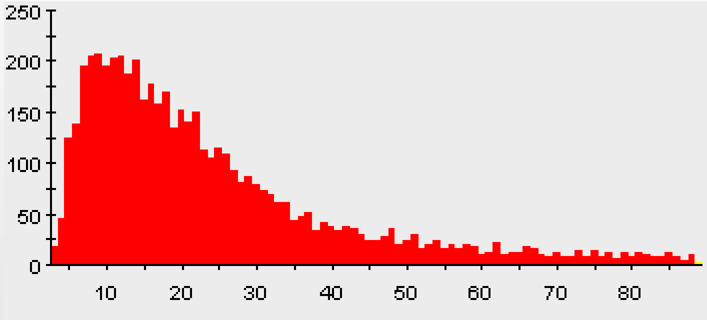
Use of circular-centred top-hat



Angle image

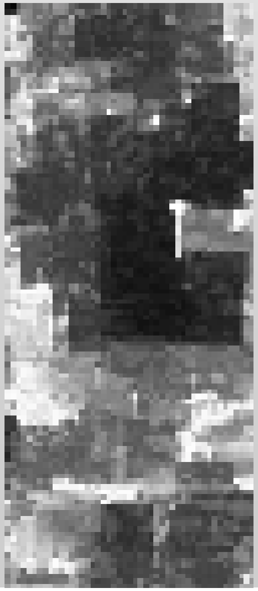
5x5

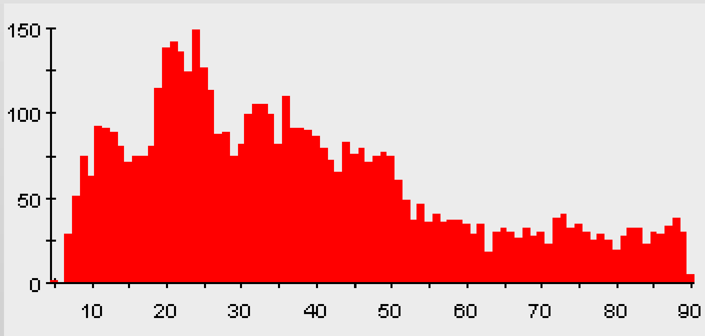


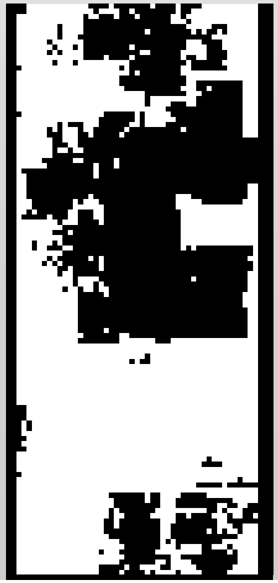


Angular top-hat and histogram

Threshold 30-90



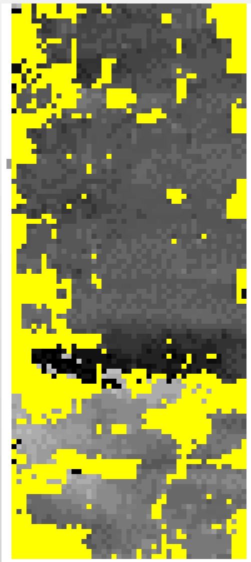


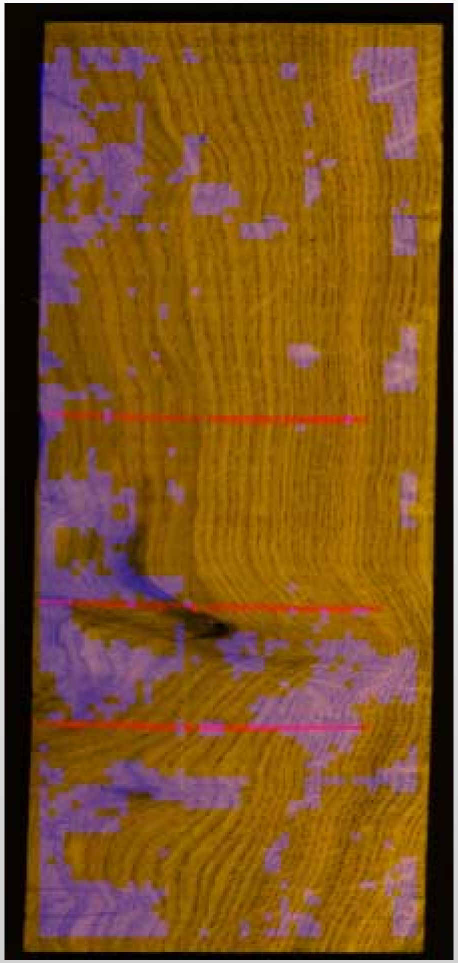


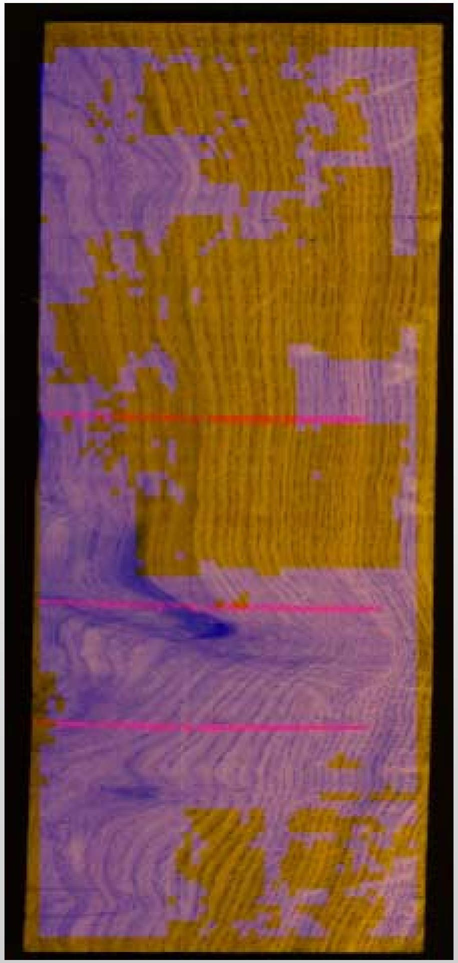
9x9

Use of circular-centred top-hat

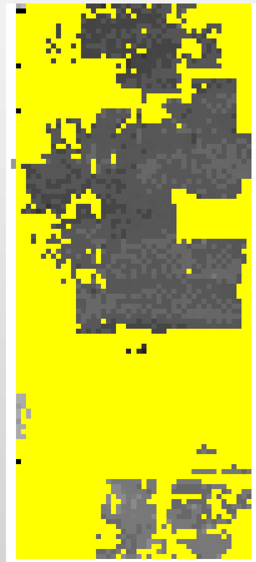
5x5







9x9



Summary

- Using mathematical morphology on angle valued functions is difficult
- To combat this, we have developed rotationally invariant operators
- Two possible approaches have been presented, namely
  - Circular centred operators (operators which bring into play a difference)
  - Indexed Partitions
- Applications of these operators to common angle images, the hue image and directional texture images, were presented