Frontiers of Mathematical Morphology
April 17 - 20 , 2000
Strasbourg, France

Viscous Lattices

Definition

Connections

geodesic Reconstruction

Application to watershed

By J. Serra

Ecole des Mines de Paris

Bibliography

- ***Initial Idea :***

F.Meyer, La viscosité en ligne de partage des eaux. Note CMM 1994

- ***Applications :***

1- Ph. Degrize, Reconstruction d'images IRM ; Analyse automatique d'images par Morphologie Mathématique. Phd Thesis, Université Paris 7, 1994 ;

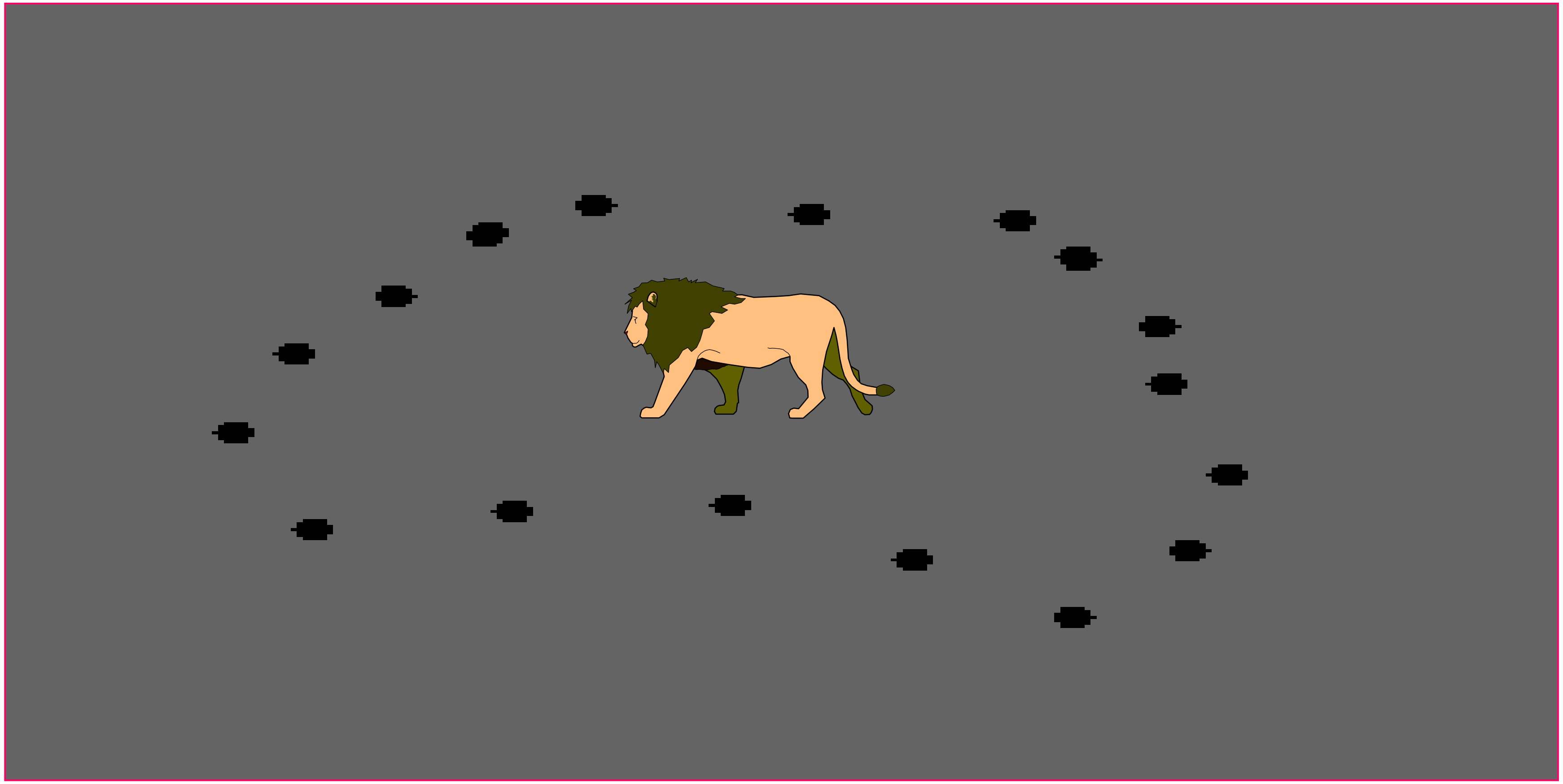
2- C. Vachier, F. Meyer et R. Lamara : Segmentation d'image par simulation d'une inondation visqueuse. To be published in RFIA 2000 - Paris ;

- ***Current Development :***

J. Serra , Viscous Lattices Note CMM D01-2000.

Objectives (1)

Purpose **:** ***To swell the space so that the Lion be wedged in its cage .***



Objectives (2)

- ***Method :***

  To generate ***viscous*** reconstruction, *i.e*. with a given meniscus.

- ***Means :***

1- To build the convenient lattice Λ : it is called viscous, and generated by the dilates of the points by a given dilation δ (the viscosity) ;

2- To define connections on Λ and to use the associated connected openings.

- ***An example :*** Contour of the heart muscle.

Notation and Reminder

- ***Notation :***

E : an arbitrary set ; Π(E): the lattice of all subsets of E;

δ : dilation Π(E) → Π(E), of adjoint erosion ∂. Dilation δ is determined by the images of the singletons {↵} de Π(E) :

$$\delta(X) = \frown\{\,\delta(↵),\ ↵ \varepsilon X\,\} \qquad X\,\varepsilon\,\Pi(E) \qquad (1);$$

B = {δ(↵) , ↵ ε E} : class of the dilates of the singletons;

γ = δ ⌊ ∂ = opening adjoint to dilation δ.

- ***Reminder :*** *The family* Λ = { δ(X), X ε Π(E)} *of the dilates of the elements of Π(E) is also the image of Π(E) under the opening γ = δ ∂, adjoint to dilation δ.*

Viscous Lattices

- ***Proposition 1 :***

  Set Λ has a structure of **complete lattice** *for the ordering of the inclusion. In this lattice, the supremum coincides with the set union, although the infimum $\wedge$ is the opening of the intersection by $\gamma = \delta\,\partial$*

$$\wedge\{X_i\,,\, i\varepsilon I\} = \gamma(\cup\{X_i\,,\, i\varepsilon I\}) \qquad \{X_i\,,\, i\varepsilon I\}\,\varepsilon\Lambda \quad (2)$$

- *The extreme elements of Λ are E and the empty set $\emptyset$.*
- *set Λ is said to be the* ***viscous lattice of dilation δ***.

Atoms et Sup-generators

- ***Sup-generators :***

  The class B of the singletons dilates is a sup-generator of lattice Λ .

- ***Atoms :***

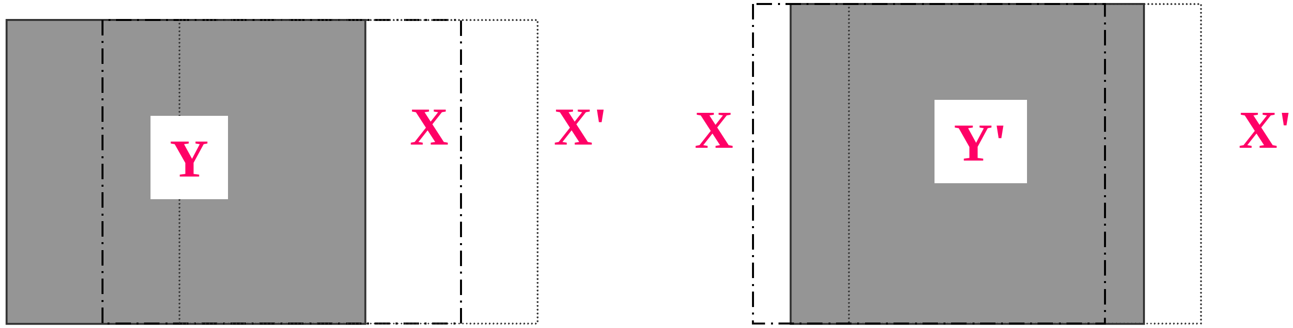
  But the $\delta(\lrcorner)$ themselves are not atoms in general. However, when $E = \nabla^n$ or $\wedge^n$, and for δ invariant under translation, the elements of B are the translates of the transform $B = \delta(0)$ of the origin.

  In such a case, and for B a compact set, the associated viscous lattice is atomic, of atoms the translates de B .

A few Counter-performances (1)

- ***Proposition 2 :***
  The viscous lattice of dilation δ is generally neither distributive, nor co-prime, and does not admit unique complements.

- ***An example of the lack of distributivity:***



$$(X \wedge X') \cap Y = \emptyset \cap Y \neq (X \cap Y) \wedge (X' \cap Y) = (X \cap Y)$$

$$(X \cap X') \wedge Y' = Y' \neq (X \wedge Y') \cap (X' \wedge Y') = \emptyset$$

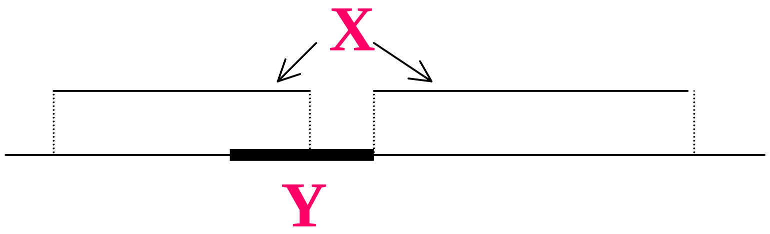
A few Counter-performances (2)

- ***Complement :***

A complement to X ε Λ is every set Y ε Λ such that

$$\mathbf{Y \quad X^c \quad \text{and} \quad \gamma(Y \cup X) = \varnothing}$$

- ***An example :***



- ***What is left :***

is Galois involution between adjoint dilation and erosion, i.e

$$\mathbf{\delta(Y) \subseteq X \quad \Leftrightarrow \varepsilon(X) \subseteq Y \qquad\qquad X, Y \,\varepsilon\, \Lambda \qquad\qquad (3)}$$

Dilation and Erosion in Λ

- ***Links between a dilation viewed in Λ and in $\Pi(E)$:***

Dilation ***erosion***

$\alpha : \Pi(E) \rightarrow \Pi(E)$ $\qquad$ $\alpha^{-1} : \Pi(E) \rightarrow \Pi(E)$

$\alpha : \Lambda \rightarrow \Lambda$ $\qquad$ $\beta : \Lambda \rightarrow \Lambda$

Identity of α acting in $\Pi(E)$ or in Λ...

- ***Links between an erosion dilation viewed in Λ and in $\Pi(E)$:***

$$\beta(X) = \cap\{\, \delta(\hookleftarrow) ,\ \delta(\hookleftarrow) \subseteq \alpha^{-1}(X) \,\} = \beta(X) = \delta\partial\alpha^{-1}(X)$$

When α and δ commute, then erosion β , in Λ , equals the opening by $\gamma = \delta\partial$ of erosion α^{-1}, adjoint to dilatation α in $\Pi((E)$.

Connection (Reminder)

- ***Set case :***

Every set family $X \subseteq \Pi(E)$ that satisfies the 3 following axioms

i / $\emptyset \varepsilon X$

ii/ ↲ ε E $\Rightarrow$ {↲} ε X (class X is sup generating)

iii/ $\{ X_i, i \varepsilon I\} \varepsilon X$ et $\cup X_i \neq \emptyset \Rightarrow \cap X_i \varepsilon X$ (class X is conditionally closed under union).

- ***Generalisation :***

The definition of connectivity extends to any sup generating lattice, by replacing $\emptyset$ by the zero of the lattice, and the set $\cap$ and $\cup$ by the sup et l'inf of the lattice. Axiom ii/ just means that class X is sup-generating.

First Connections on Λ(1)

- ***Definition :***

Let Λ be a viscous lattice of dilation δ on $\Pi(E)$. A class X' of Λ defines a connection on Λ when

i / $\emptyset \in X'$

ii/ $x \in E \Rightarrow \delta\{x\} \in X'$ (5)

iii/ $\{X_i, i \in I\} \in X'$ **and** $\wedge X_i \neq \emptyset \Rightarrow \cap X_i \in X'$

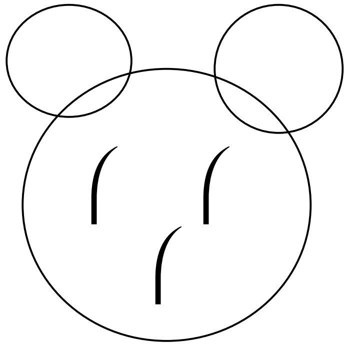
- ***Proposition 3 :***

A class $X \subseteq \Lambda$ is a connection on Λ. if it is the restriction to Λ of the union of a connection X on $\Pi(E)$, and of the image B of the singletons of $\Pi(E)$ under δ, i.e. if

$$X' = (X \cup \Lambda) \cap B = (X \cap B) \cup \Lambda$$

First Connections on $\Lambda(2)$

- ***An Example :***



X = *Arcwise connection* Λ = *set of the dilates by the unit disc B.*

The union of the three lobes belongs to both X and Λ, hence to X '. However, the three lobes are disjoint in Λ because their erosions by B are disjoint in $\Pi(E)$.

If we want to get separated particles here (for connections on Λ), we have to take into account the status of connectivity (in $\Pi(E)$) ***before*** dilation by δ.

Second Connections on Λ(1)

- ***Theorem 1 :***

Let X be a connection on $\Pi(E)$ and $\delta : \Pi(E) \rightarrow \Pi(E)$ be a dilation, of adjoint erosion ∂ , which generates the viscous lattice Λ. Then the image $X' = \delta(X)$ of connection X turns out to be a connection on Λ .

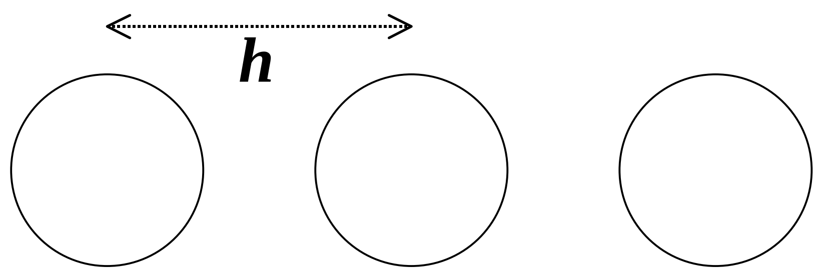
- ***Comments :***

- these new connections on Λ, are sensibly more restrictive than those of prop. 4 .
- For example, the three lobes of the figure are now three *disjoint* connected components.

Second Connections on Λ(2)

- ***Generality of the theorem :***

- We did not assume that the dilates of the elements de X are still X-connected. For example, in ∇^2, with the usual arcwise connection, take for δ the dilation by a doublet of points from h apart. Then the left two discs of the figure form a connected set, although the group of the three discs is no longer connected



- However, if ↵ε E ⇒ δ {↵}ε X, we have X' = δ (X) ⊆)X and the elements of X' are then connected in both lattices Λ and Π(E) : Dilation δ **preserves** connection X .

Geodesic Operators (1)

- ***Theorem 2 :***

  When dilation δ preserves connection $\mathcal{X}$, then connection $\mathcal{X}$ ' induced on Λ , in the sense du theorem 1, is also preserved by δ .

  In addition, if $\gamma_{\hookleftarrow}$ et $\gamma'_{\delta(\hookleftarrow)}$ stand for the elementary connected openings on $\Pi(E)$ and on Λ respectively, then

$$\gamma'_{\delta(\hookleftarrow)} = \delta\partial\gamma_{\hookleftarrow} = \gamma_{\hookleftarrow}\delta\partial \qquad \hookleftarrow \in E .$$

- ***Comment :***

  Every X'-particle is the opening by $\delta\partial$ of the corresponding X-particle. That allows to extract the X'-particles.

  Alternatively, can we build **directly** geodesic dilations in Λ ?

Geodesic Operators (2)

- ***Proposition 4 :***

Let dilation δ be extensive and preserving connection $X \subseteq \Pi((E)$. Given $A;Z \varepsilon$ $\Pi((E)$, with $A \subseteq Z$ and X -connected, the conditional dilate

$$\zeta(A) = \delta(A) \wedge Z$$

is X '- connected and included in Z.

- ***N. B. :*** This does not mean that the iterated versions of ζ tend to a X '- component of Z . Here is a counter-example in ∇^2 :

E := square D of side a and of centre y ;

B(y) := disc of diameter < a centred in y

Dilation := $\delta(\downarrow) = \{\downarrow\}$ $\downarrow \varepsilon$ D / y ; $\delta(y) = B(y)$

Geodesic Operators (3)

- ***Definition:***

Dilation δ is said to be **completely extensive** *when for any point ↲ ε E (E topological space) and for any compact set $K \subseteq E$, there exists an integer n such that the n^{th} iteration of δ(↲) covers K.*

- ***Proposition 5 :***

When dilation δ is completely extensive, then the X'-connected component Z of Λ marked by $A \,\varepsilon\, X'(\Lambda)$, if it can be covered by a compact set Z_0 of $\Pi(E)$, is equal to

$$\mathbf{Z = \zeta^{(n)}\ (A)}$$

for some integer n.

An Example of Binary Geodesy (1)

a) *b)* *c)*

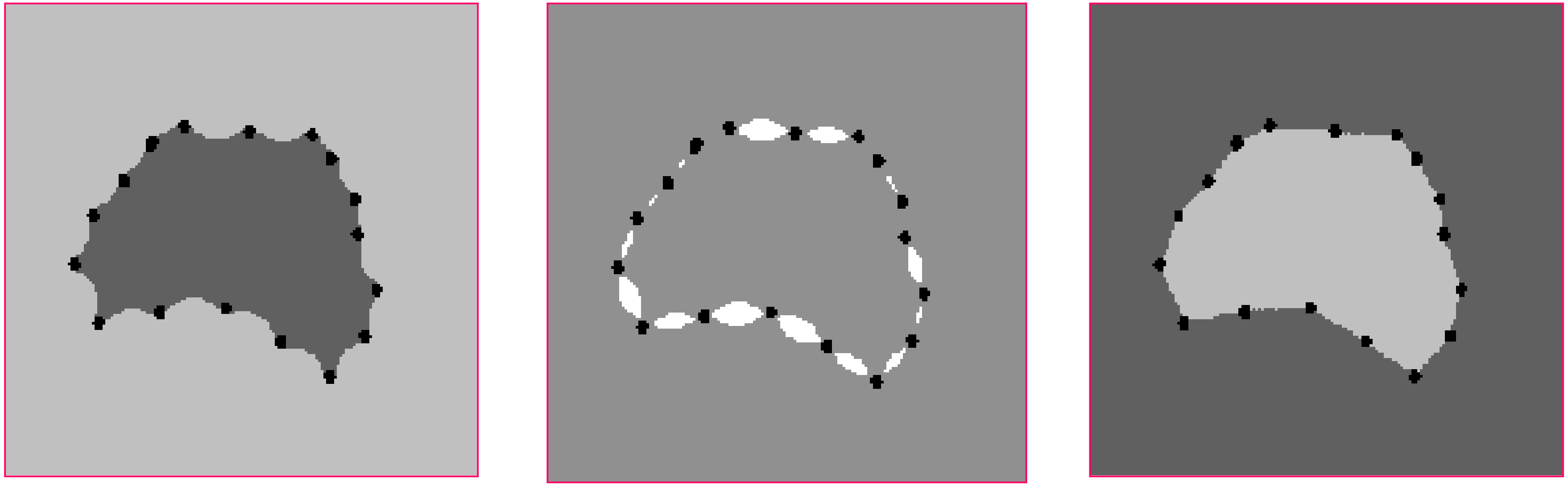
Binary example : in white, marker A ; in black, complement of mask Z; dilation δ is the Minkowski addition by the disc of radius r :

a) step of reconstruction for r = 15 ;

b) reconstruction for r = 27 ;

c) maximum reconstruction, corresponding to r = 17 .

An Example of Binary Geodesy (2)



a) *b)* *c)*

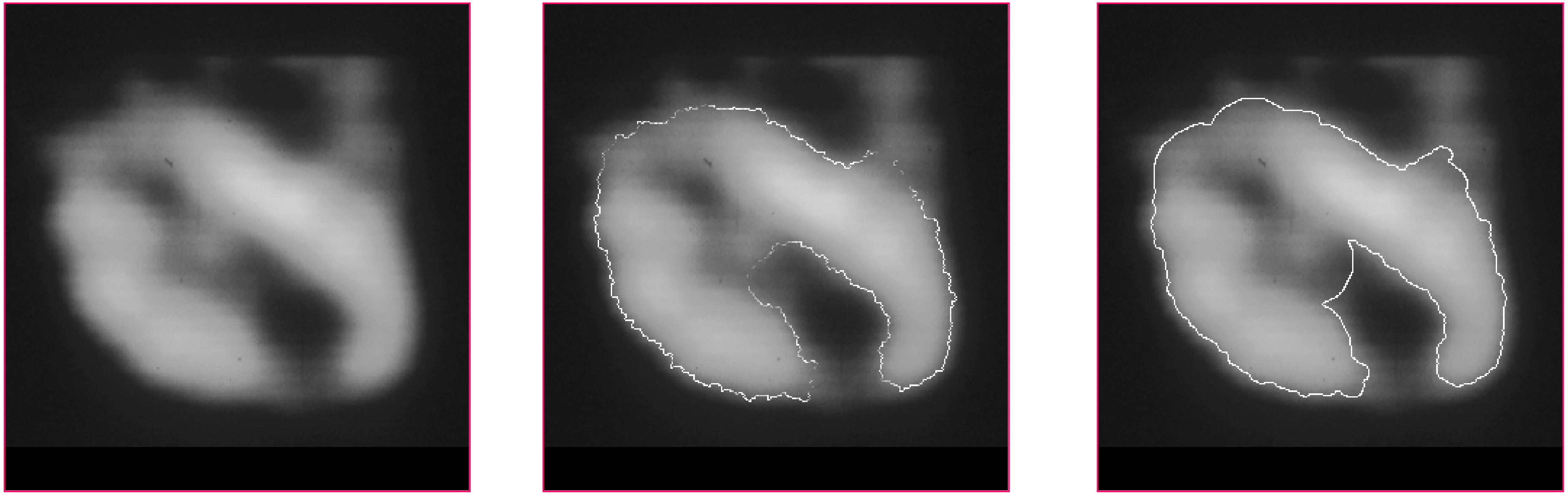
Binary example (followed) :

a) maximum reconstruction from the edges of the field

b) partition of Z into two X'-connected components for δ_{max}

c) corresponding median element .

An Example of Numerical Geodesy (1)



a) *b)* *c)*

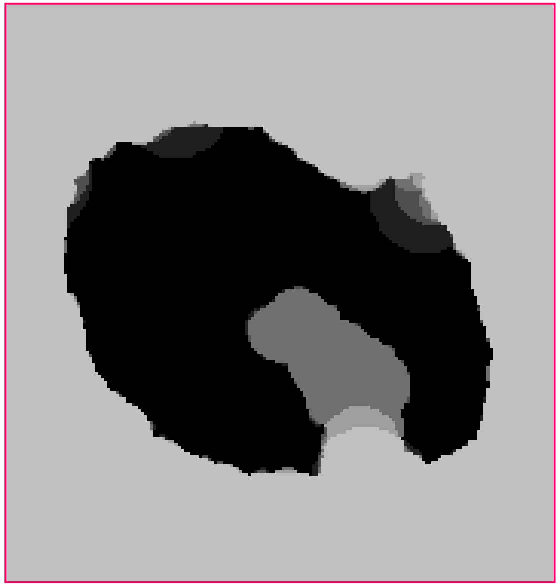
Numerical example :

a) positron image of the heart muscle (copyright CEA-ARMINES) ;

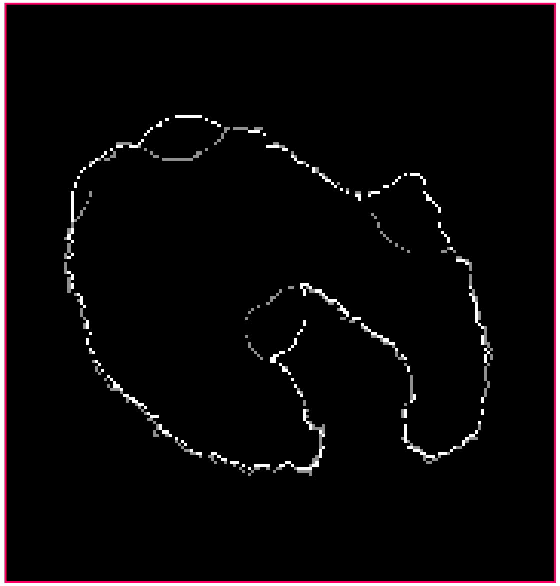
b) Watershed line of the gradient of fig.a

c) optimum reconstruction of the X'-components internal to zone b.

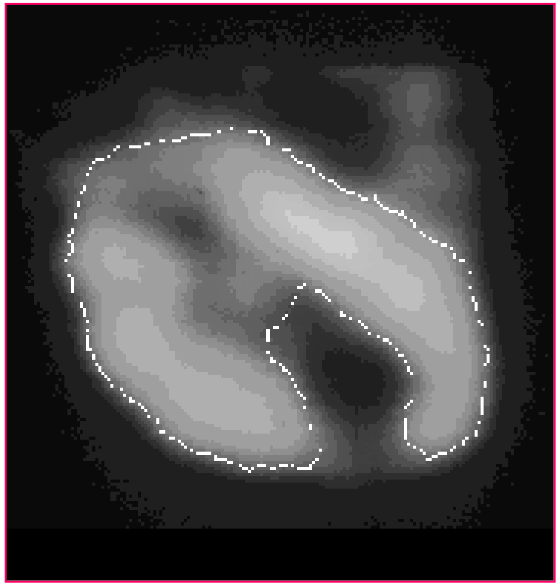
An Example of Numerical Geodesy (2)



a)



b)



c)

Numerical example (followed) :

a) external surface of maximum viscosity ;

b) external and internal contours ;

c) median element between the two contours.